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THE THIRTY-THIRD SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-third summer meeting and eleventh colloquium of the Society was held at the University of Wisconsin, Madison, Wisconsin, from Tuesday to Saturday, September 6–10, 1927, preceded by the summer meeting of the Mathematical Association of America. It will be remembered as one of the most notable occasions in the annals of the Society.

The colloquium lectures by Professors Bell and Wheeler were delivered on Tuesday, Wednesday, Thursday, and Saturday mornings, and Thursday evening. On Thursday afternoon and Friday morning the Society met in sections for the reading of papers, the Sections of Analysis and of Geometry meeting on Thursday and those of Algebra and Analysis and Miscellaneous Subjects on Friday. The joint dinner of the Society and Association with Professor C. S. Schlichter as toastmaster held at the Maple Bluff Golf Club on Wednesday evening was attended by 150 persons.

A reception was tendered the visiting mathematicians and their friends by the officials of the University, and a program of interesting excursions was carried out, and ample provision was made for sports. A picture of the attending mathematicians and their friends was taken, a copy of which is being distributed with this issue of the Bulletin.

The attending mathematicians and their friends were entertained at Chadbourne and Barnard Halls of the University; the meetings were held in Bascom Hall and the colloquium lectures in Sterling Hall.

Resolutions were passed expressing profound appreciation of the cordial hospitality of the University; thanks to the department of mathematics and in particular to the local members of the Committee on Arrangements, Professors E. B. Van Vleck (Chairman), Arnold Dresden, and Warren Weaver; to Professor R. W. Babcock, who was in

charge of the registration at Chadbourne Hall; and to others who contributed to the success of the gathering.

Many attending members recalled with pleasure the earlier summer meeting and colloquium held at the University of Wisconsin in 1913. On that occasion Professor L. E. Dickson gave a series of lectures entitled "On invariants and the theory of numbers," and Professor W. F. Osgood a series entitled "Topics in the theory of functions of several complex variables." The 1927 colloquium must therefore be known as the Second Madison Colloquium. A special report is given on page 663 of the present issue of the Bulletin.

The attendance included the following one hundred thirty-three members of the Society:

F. E. Allen, Altshiller-Court, W. E. Anderson, Archibald, R. W. Babcock, Barnett, Batchelder, W. O. Beal, E. R. Beckwith, E. T. Bell, Birkhoff, Blichfeldt, Bray, Brenke, Brink, E. W. Brown, H. E. Buchanan, L. H. Bunyan, Bussey, Cairns, C. C. Camp, Carlson, Carmichael, Chang, Chittenden, J. T. Colpitts, Comstock, Conkwright, C. F. Craig, H. V. Craig, Crathorne, Curry, Dalaker, H. T. Davis, Dean, Dickson, Dines, Dowling, Dresden, Emch, G. C. Evans, H. P. Evans, Feinler, W. B. Ford, Fry, Gaba, Garrett, Gergen, Gill, D. C. Gillespie, M. C. Graustein, W. C. Graustein, L. M. Graves, Griffiths, V. G. Grove, Guggenbühl, Harkin, W. L. Hart, W. W. Hart, Hartung, E. R. Hedrick, Hildebrandt, Hoersch, Hollcroft, Hosford, Hotelling, J. C. Hughes, Louis Ingold, Ingraham, Dunham Jackson, Jensen, Kearney, Kempner, Krathwohl, E. P. Lane, Langer, Latimer, Harry Levy, Logsdon, J. V. McKelvey, MacMillan, March, Maria, A. S. Merrill, Michal, Mickelson, Mills, Miser, E. J. Moulton, Olson, F. W. Owens, H. B. Owens, Palmer, Parkinson, Pettit, E. C. Phillips, Phipps, Pierce, Pogo, R. G. D. Richardson, H. L. Rietz, E. D. Roe, Roever, Roos, Roth, Runge, Sanger, Shewhart, Shohat, Sinclair, Skinner, Slaught, Slichter, Snedecor, Virgil Snyder, Stouffer, Swartzel, Tate, J. H. Taylor, E. M. Thomas, Torrey, Trjitzinsky, Van Vleck, Wall, J. H. Weaver, Warren Weaver, Westfall, A. P. Wheeler, Widder, Wolff, Worthington, Yanney, Yeaton.

The Secretary announced the following elections to membership in the Society:

To sustaining membership:

The Georgia Power Company, Atlanta, Ga.;
The University of Michigan, Ann Arbor, Mich.

To ordinary membership:

Dr. Clement Walker Andrews, Librarian, John Crerar Library;
Mr. Loring Beal Andrews, Harvard University;

Mr. John William Arnold, Western Union Telegraph Co., New York City;
Professor Herman T. R. Aude, Colgate University;
Mr. Benjamin Brownstein, tube mill engineer, Ellwood City, Pa.;
Mr. Arnold Chaimovitch, University of Chicago;
President Harry F. Chapell, Chapell Auditors, New York City;
Professor Edward Everett Colyer, Kansas State Teachers College, Hays;
Professor Walter H. Durfee, Hobart College;
Mr. Theodore Miller Edison, East Orange, N.J.;
Mr. Henry Ericson, Washington High School, Milwaukee;
Mr. John Graham Foley, engineer, New York Central Railroad;
Mr. Edwart T. Frankel, National Industrial Conference Board, New York City;
Mr. Albert R. Gallatin, of Smith and Gallatin, New York City;
Professor Joaquin Gallo, Observatorio Astronomico Nacional, Tacubaya, Mexico;
Professor Horace Newton Hubbs, Hobart College;
Mr. Eugene Law Ickes, geologist, Los Angeles;
Professor Richard P. Johnson, Carnegie Institute of Technology;
Miss Elizabeth E. Knight, Milwaukee State Normal School;
Mr. Joseph Harrison Kusner, University of Florida;
Mr. John Buchanan MacLeod, Chicago;
Dr. James D. Maddrill, Union Labor Life Insurance Company, Washington;
Mr. Phillip Newman, Columbia University;
Mr. William Ross Paul, Chicago;
Professor Thomas Reeve Rosebrugh, University of Toronto;
Mr. Samuel Barnell Rosenbaum, Milford School, Milford, Conn.;
Professor Anestis George Sarafoglou, Robert College, Constantinople;
Dr. Gyan Chand Sharma, University of Notre Dame;
Mr. Hartley L. Smith, New England Power Company, Worcester, Mass.;
Professor Louis Beaufort Stewart, University of Toronto;
Dr. Dirk Jan Struik, Massachusetts Institute of Technology;
Mr. Paul McCartney Swingle, Ohio State University;
Miss Helen Thompson, Psycho-Clinic, Yale University;
Mr. James Edgar Thompson, Pratt Institute;
Mr. Charles Wexler, Roxbury, Mass.;
Mr. George Washington Wishard, Norwood, Ohio;
Mr. Byron C. Wolverton, Friendship Telephone Company, Syracuse;
Mr. Samuel Harry Woolf, Harvard University.

Nominees of sustaining members:

Mr. Charles M. Biscay, Western and Southern Life Insurance Company;
Mr. Charles M. Biscay, Jr., Western and Southern Life Insurance Company;
Mr. Wyman A. Bristol, University of Pennsylvania;
Mr. D. K. Chenoweth, Western and Southern Life Insurance Company;
Mr. C. E. Nelson, Missouri State Life Insurance Company;
Mr. R. A. Ryan, Western and Southern Life Insurance Company;
Professor S. E. Stillwell, Western and Southern Life Insurance Company.

The following members of the London Mathematical Society have accepted membership under the reciprocity agreement:

Professor William E. H. Berwick, University College, Bangor, Wales;
Mr. Charles Henry Rowe, fellow of Trinity College, Dublin.

Twenty-three applications for membership were received.

On recommendation of the Council, the Society adopted two amendments of the by-laws. The first adds to the membership of the Council the three representatives of the Society on the Editorial Board of the American Journal of Mathematics, thus placing this periodical on the same basis as the Bulletin and the Transactions. The second replaces the Assistant Secretary by two Associate Secretaries, with the expectation that the extra official will be in the east.

It was announced that Professor E. W. Brown has accepted the invitation to give the fifth Josiah Willard Gibbs Lecture in connection with the 1927 Annual Meeting in Nashville.

The following appointments were announced: Committee on the Society Visiting Lectureship, Professors G. D. Birkhoff, G. A. Bliss and E. R. Hedrick; Committee on Arrangements for the Gibbs Lecture, Professors W. L. Miser (Chairman) and James McClure of Vanderbilt University and Mr. J. L. Wilson of Nashville; additional representatives of the Society on the augmented American Section of the International Mathematical Union, Professors G. A. Bliss, A. B. Coble, Arnold Dresden, E. V. Huntington, and C. H. Sisam.

On account of the change in his residence Professor Arnold Dresden presented his resignation as Assistant Secretary and a resolution of thanks was passed by the Society as follows:

"Resolved that we, the members of the American Mathematical Society, especially those who have been attending the Chicago or Western meetings of this Society, express to Professor Arnold Dresden our sincere appreciation of and gratitude for the many ways in which he has been of service to these meetings during the last eleven years. We feel that

his genuine interest in the welfare of the Society, his untiring efforts in its behalf, and the consummate skill with which he has conducted its affairs, including the arrangements for its meetings, in this section of the country have been of inestimable value to the growth of mathematical interest."

Professor M. H. Ingraham was appointed to fill the vacancy for the remainder of 1927.

Titles and abstracts of the papers read before sectional sessions of the Society follow below: the papers numbers 1 to 8 were read before the Section of Analysis, Professor G. C. Evans presiding; numbers 9 to 16 and 69 before the Section of Geometry, President Snyder presiding; numbers 17 to 30 and 66 to 68 before the Section of Algebra, Vice-President Bell presiding; and numbers 31 to 65 before the Section of Analysis and Miscellaneous Subjects, ex-president Birkhoff presiding. Mr. Heineman was introduced by Professor Dresden, Professor Besicovitch by Professor Tamarkin (his paper was read by Professor Langer), Professor Uspensky by the Secretary, and Mr. Dorroh by Professor R. L. Moore. The papers of the following authors were read by title: Ayres, Birkhoff (first paper), Buchanan (second paper), Evans (second paper), Kempner (first paper), Langford, Mears, Merriman, Miller, Moore, Rainich, Schelkunoff, Sharpe, Roos (second paper), Uspensky (second paper), Vandiver, Walsh, Webber, G. T. Whyburn, W. M. Whyburn, Widder and Gergen, Wilson.

1. Professor J. A. Shohat: *On certain developments in series of Tchebycheff polynomials.*

Consider the finite development $x^n = H_0\phi_0(p; x) + \cdots + H_n\phi_n(p; x)$, where $\phi_n(p; x)$, $n = 0, 1, 2, \cdots$, is a system of Tchebycheff polynomials corresponding to a given interval (a, b) with the characteristic function $p(x)$. In the case $(a, b) = (-1, 1)$, $p(x) \equiv 1$ (Legendre polynomials), all the coefficients H_i above, as is well known, are positive, a property leading to many interesting results. In the present paper the author extends this property of the H's to the general case of "symmetric Tchebycheff polynomials": $a = -b$ (finite or infinite), $p(x) \equiv p(-x)$. Application is made, in particular, to the discussion of the convergence of the development $f(x) \backsim \sum_{n=0}^{\infty} A_n\phi_n(p; x)$, where the $A_n \equiv \int_{-a}^{a} p(x)\phi_n(p; x)f(x)dx$ are all positive and $f(x)$ and $p(x)$ are subject to certain general conditions.

2. Professor L. L. Dines: *Linear inequalities in general analysis.*

This paper appears in full in the present number of this Bulletin.

3. Professor L. L. Dines: *A theorem on orthogonal sequences.*

For any positive constant e greater than unity, we denote by $\mathfrak{S}_e'$ and $\mathfrak{S}_e''$ the classes of all infinite real sequences $\{\sigma_i'\}$ and $\{\sigma_i''\}$ respectively, such that the series $\sum_i|\sigma_i'|^e$ and $\sum_i|\sigma_i''|^{e/(e-1)}$ converge. If $\{\sigma_i'\}$ and $\{\sigma_i''\}$ are sequences of these respective classes, then the series $\sum_i\sigma_i'\sigma_i''$ converges absolutely, as is well known. If the sum of this product series is zero, the two sequences are said to be *orthogonal.* We prove the following theorem: A necessary and sufficient condition that there exist in $\mathfrak{S}_e''$ a positive sequence orthogonal to each of a given finite set of sequences in $\mathfrak{S}_e'$ is that no linear combination of the given set be M-definite.

4. Professor R. E. Langer: *The boundary problem associated with a differential equation in which the coefficient of the parameter changes sign.*

This paper deals with the boundary problem associated with the differential system $u''(x)+\lambda x^\nu u(x)=0$, $u(-\alpha)=u(\beta)=0$, $\alpha, \beta>0$, in which ν is any real positive constant. The interest in the system centers on the fact that the interval considered contains the origin, so that the coefficient of the parameter changes sign if ν is an odd integer, and in general changes from real to complex. The method, that of asymptotic forms, has not heretofore been applied to the study of systems of this type. Asymptotic expressions for the characteristic values and functions are derived, and the theorem for the expansion of an arbitrary function as established is far more general than the existing theorems derived by other methods.

5. Professor W. B. Ford: *On the behavior of integral functions in distant portions of the plane.*

This paper will appear in full in an early number of this Bulletin.

6. Professor Dunham Jackson: *On the approximate representation of analytic functions.*

This paper will appear in full in an early number of this Bulletin.

7. Professor T. H. Hildebrandt: *Lebesgue integration in general analysis.*

This paper points out a method for defining a Lebesgue type of integration in the case of functionals on a linear real range to a complete linear function space. Riemann integration for such functionals has been defined by L. M. Graves (Transactions of this Society, vol. 29 (1927), pp. 166 ff.). The method of defining a Lebesgue integration is suggested by the validity

of Osgood's theorem in this situation, i.e., if a sequence of Riemann integrable functionals bounded in their totality converges to a Riemann integrable functional, then the limit of the integrals is the integral of the limit. Starting with a class of Riemann integrable functionals, e.g., the continuous functionals, and using convergent (or almost convergent) sequences of functionals, we can define integration for functionals not Riemann integrable.

8. Professor T. H. Hildebrandt: *Note on interchange of order of limits.*

This paper will appear in full in an early number of this Bulletin.

9. Mr. G. A. Parkinson: *A theory of parallelism in sub-spaces.*

In this paper the angle between two vectors at different points of a V_n is defined. On the basis of this definition, the notion of the parallel motion of a vector in V_{m_1}, a sub-space of V_n, with respect to V_m, also a sub-space of V_n, is developed. The special case in which V_{m_1} and V_m are applicable is discussed. Parallel curves of V_n are defined, and a notion of orders of parallelism between sub-spaces V_{m_1} and V_m is discussed.

10. Mr. S. A. Schelkunoff: *On certain properties of orthogonal and generalized orthogonal transformations.*

This paper gives an extension of Euler's formulas for transformation of coordinates, and certain properties of the coefficients, also a decomposition of a general rotation in an n-flat into a succession of "simple" rotations, and the determination of the angles of simple rotations. The generalized orthogonal transformations are defined by means of relations between the coefficients a_k^i: $a_k^i \bar{a}_k^j = 1$ when $i = j$, and $a_k^i \bar{a}_k^j = 0$ when $i \neq j$, where $\bar{a}_k^i$ is the conjugate of a_k^i. Certain properties of generalized orthogonants are then obtained. The remainder of the paper is concerned with quasi-metrical transformations of null-flats.

11. Professor T. R. Hollcroft: *Limits for double points of surfaces.*

Limits that can be reached but not exceeded are found for the number of distinct conic nodes of an algebraic surface of any order. For orders greater than seven, these limits are smaller than those previously found. Also the maximum number of binodes of each kind and unodes of each kind is found for any algebraic surface. It is further shown that all double points of whatever kind that an algebraic surface can have may be real. The maximum numbers of consecutive conic nodes, binodes and unodes of surfaces of given order are determined. Distinct conic nodes account for one invariant each, but when they combine to form a higher singularity, the number of conic nodes is greater than the number of invariants. A certain number of conic nodes may thus be considered independent and the rest dependent. For a given order of a multiple point, a limiting order

of the surface containing it is found such that for a smaller order of the surface the number of independent conic nodes in the multiple point exceeds the maximum number of distinct conic nodes a surface of that order can have. The existence of the surfaces is shown to depend upon that of plane curves with distinct double points.

12. Professor Nathan Altshiller-Court: *On four mutually orthogonal circles.*

Four mutually orthogonal circles are considered in their relations to the triangles formed by their centers and to the circumcircles of these triangles. Among others the following results are obtained. The square of the radius of any one of four given mutually orthogonal circles is equal to one half of the power of its center with respect to the circle passing through the centers of the remaining three circles. The algebraic sum of the squares of the radii of the given circles is equal to the square of the diameter of the circle passing through the centers of three of the given circles. Each of the given circles is the conjugate circle of the triangle formed by the centers of the remaining three circles, and, conversely, the conjugate circles of an orthocentric group of four triangles are mutually orthogonal. The trilinear pole of an axis of similitude of three of the given circles with respect to the triangle formed by their centers coincides with the pole of this line with respect to the fourth circle.

13. Professor J. H. Weaver: *On certain points associated with a triangle.*

The author has discussed the properties of four points associated with a triangle and defined as follows. Let triads of circles be drawn in such a way that the internal segments described on BC, CA, AB respectively contain angles (1) C, A, B; (2) B, C, A; (3) $2C$, $2A$, $2B$; (4) $2B$, $2C$, $2A$. These four triads of circles determine four points, P_1, P_2, P_3, P_4. Among the interesting properties of these four points the following may be noted: (1) the pedal triangles of the four points are similar; (2) the pairs of lines BP_1 and CP_2, CP_1 and AP_2, AP_1 and BP_2 are parallel; (3) if BP_2 and CP_1 intersect in A_1, CP_2 and AP_1 intersect in B_1, and BP_1 and AP_2 intersect in C_1, then the triangles ABC and $A_1B_1C_1$ are congruent and have the sides AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 respectively parallel; moreover, the points P_1 and P_2 bear the same relation to each of the triangles ABC and $A_1B_1C_1$; (4) if the points P_3 and P_4 replace P_1 and P_2 in (3), then three points A_2, B_2, C_2 are determined. These three points and the points P_3 and P_4 lie on a circle.

14. Professor Harold Hotelling: *Manifolds of orthogonal ennuples.*

Regarding an orthogonal ennuple as a set of n mutually perpendicular non-sensed and indistinguishable lines through a point in n-space, the set of orthogonal ennuples at a point constitutes a manifold of $n(n-1)/2$ dimensions. This paper deals with certain analysis situs properties of these manifolds.

15. Professor F. R. Sharpe: *Superabundant nets of plane curves.*

In this paper, the author considers a general method of finding superabundant nets of plane curves which determine, by their intersections, involutions of the order of the grade. The involutions are first found on a rational surface F_n, by means of its intersections with the lines of a rational congruence C, having a fundamental line l. If l is r-fold on F_n, the involution is of order $n-r$. A net of surfaces N can be found such that any two surfaces of N meet in a line of C. The surfaces of N meet F_n in a net of curves. When F_n is mapped on a plane, the corresponding net of curves determines an involution of order $n-r$. By considering surfaces of orders 3 and 4, various new types of involutions of order 3 can be found, and similarly from surfaces of orders 4, 5, and 6, involutions of order 4.

16. Professor V. G. Grove: *Nets with equal W invariants.*

Nets with equal W invariants the author has called I nets. A conjugate I net is an isothermally conjugate net. By the introduction of the term "equal point invariants of the second kind," and by extending the idea of equal point invariants, many theorems on isothermally conjugate nets become true for I nets. Conjugate nets have equal point invariants of the second kind.

17. Professor G. A. Miller: *Substitutions which transform a regular group into its conjoint.*

This paper appears in full in the present number of this Bulletin.

18. Professor C. G. Latimer: *A note on a theorem of Hermite.*

If $|a_{ij}|$, $(i, j, =1, 2, 3)$, is the general symmetric third-order determinant, and A_{ij} is the cofactor of a_{ij}, Hermite showed that the form $\phi(x)=x_0^2+\sum_{i,j=1}^{3}A_{ij}x_ix_j$ repeats under multiplication. It may be shown that by a unitary transformation T, ϕ may be transformed into $\psi=x_0^2+bcu^2+acv^2+abw^2$, where a, b, c are rational functions of the a_{ij}. Then by use of a recent result of Bell's on the product of two such forms, and by application of the transformation T^{-1}, we obtain Hermite's formulas for the product $\phi(x)\cdot\phi(y)$. Our proof with that of Bell's result referred to above is substantially shorter than Hermite's original proof or that of Bachmann.

19. Professor C. G. Latimer: *On the representation of integers by certain indefinite quaternary forms.*

In this paper it is shown that if α is a positive integer $\leqq 163$ in the form $4k+3$, or the double of such an integer, and contains no square factor, then the form $f(x, y, z, w)=x^2+y^2-\alpha z^2-\alpha w^2$ represents all integers. The proof is by the method of descent, being a generalization of a method used by Dickson in obtaining similar results for the case $\alpha=-7$. It is shown

that if $f(X, Y, Z, W)=mn\neq 0$, then m is represented by f if the same is true of n. Repeated application of the above leads to the cases $n=\pm 1$, which are easily treated. If m is a prime, the existence of integers $X, Y, Z, W, n\neq 0$ such that $f(X, Y, Z, W)=mn$, is well known. Since f repeats under multiplication, and since f represents -1, for each of the above values of α, as may be readily verified, our result follows.

20. Professor L. E. Dickson: *All positive integers are sums of values of a quadratic function of x.*

This paper appears in full in the present number of this Bulletin.

21. Professor L. E. Dickson: *Generalizations of the theorem of Fermat and Cauchy on polygonal numbers.*

This paper will appear in full in an early number of this Bulletin.

22. Professor L. E. Dickson: *Extended polygonal numbers.*

This paper will appear in full in an early number of this Bulletin.

23. Professor L. E. Dickson: *Generalized polygonal numbers.*

These numbers include both the polygonal and extended polygonal numbers. Hence they are the values of $e(x)$ for all positive, negative, and zero integers x. Every integer is a sum of three generalized pentagonal numbers; also of three generalized hexagonal numbers; also of four generalized octagonal numbers; etc.

24. Professor A. J. Kempner: *On the use of certain types of formulas in the theory of prime numbers.*

A number of analytic expressions are known for the representation of prime numbers (by means of the gamma function, or infinite series, or definite integrals, or repeated limiting processes). None of these formulas has so far proved to be of value in the theory of prime numbers. A simple analysis of these formulas shows that they are practically all equivalent to Wilson's theorem, which is itself no deep lying theorem. It is therefore not to be expected that the mere formulation of such theorems, unless they be accompanied by a powerful analytic method of attack, will add to our knowledge in the theory of prime numbers. Such methods of attack are not known for any of the formulas.

25. Mr. W. E. Roth: *A solution of the matric equation* $P(X)=A$.

This paper is concerned with finding, by elementary methods, such solutions of the equation $P(X)=A$ as are expressible as polynomials in A. Here $P(X)$ is a polynomial in X with scalar coefficients, and A is a square matrix of order n, not necessarily non-singular. Such solutions were found by Frobenius (Berliner Sitzungsberichte, 1896) for the case

$X^2=A$; they are given by Dickson in his *Modern Algebraic Theories*, for the case $X^m=A$. In both cases, A is supposed to be non-singular. The equation $X^m=A$ has also been treated by Cecioni (Annali della Reale Scuola Normale, Pisa, 1909), and by Kreis (Vierteljahrschrift, Zurich, 1908); in his Zurich dissertation, the latter author had also studied the general equation $P(X)=A$. In all of the papers cited, use is made of the Weierstrass elementary divisors and associated normal forms; the method is not easy to follow and is complicated in its application.

26. Mr. E. R. Heineman: *Generalized Vandermonde determinants.*

A generalized Vandermonde determinant is obtained from the ordinary Vandermonde determinant by permitting the indices to take any set of values. The Vandermonde matrix is defined to be the Vandermonde determinant with the nth powers of its variables added as an extra row. By successively blocking out each of the first n rows of this matrix, we obtain n determinants which can be called secondary Vandermonde determinants. The ordinary Vandermonde determinant, which we get by omitting the last row of this matrix, will be called the principal Vandermonde determinant. It can be shown that every generalized Vandermonde determinant is expressible as a determinant-function of the principal and secondary Vandermonde determinants.

27. Professor H. S. Vandiver: *On the theorem of Kummer concerning power characters of units in a cyclotomic field.*

This paper presents, in a modern form, the argument used by Kummer in Crelle's Journal (vol. 44 (1852), pp. 121–130) for obtaining expressions giving the pth power characters of units in the field defined by a primitive pth root of unity, p being an odd prime. Extensive use of these results will be made in other papers by the author.

28. Professor H. S. Vandiver: *Summary of results and proofs concerning Fermat's last theorem.* (Third note.)

In this note, among other results a proof is indicated for the following theorem: Under the assumptions (1) none of the Bernoulli numbers $B_{rp}(r=1, 2, 3, \cdots, (p-3)/2)$ is divisible by p^3, and (2) the second factor of the class number of the field $k(\alpha)$, $\alpha=e^{2i\pi/p}$, is prime to p, it follows that $x^p+y^p+z^p=0$ is not satisfied in rational integers x, y, and z, prime to each other, none zero, if $xyz\equiv 0 \pmod{p}$, and p is an odd prime.

29. Professor H. S. Vandiver: *On an extension of the Bernoulli summation formula.*

In this note is considered a theorem of Glaisher's which expresses in terms of Bernoulli numbers the sum of nth powers of integers in an arithmetical progression.

30. Professor H. S. Vandiver: *On the norm-residue symbol in the theory of cyclotomic fields.*

This symbol (Hilbert, *Bericht*, p. 413) has always, apparently, been used to represent a summation. In the present paper it is shown how the summation may be carried out.

31. Professor C. C. Camp: *A multiple-parameter differential system which leads to a development of a function in p variables.*

The system $u_i' + (\sum_{j=1}^{p} \lambda_j a_{ij}) u_i = 0$, $i = 1, 2, \cdots, p$, with the boundary conditions $u_i(\pi) = u_i(-\pi)$, $i = 1, 2, \cdots, p$, where a_{ij} is an integrable function of x_i either identically zero or of constant sign and such that the determinant $|A_{ij}| \neq 0$, $2\pi A_{ij} \equiv \int_{-\pi}^{\pi} a_{ij} dx_i$, leads to a multiple expansion in which the coefficients involve the p-fold integral of $f|a_{ij}| \prod_{i=1}^{p} v_i^*$; v_i^*, $i = 1, 2, \cdots, p$, being a principal solution of the adjoint system. We make the change of parameters $\nu_i = \sum_{j=1}^{p} \lambda_j A_{ij}$ for the convergence proof, and in the first integrations by parts in the contour integrals we replace the elements of the first column of $|a_{ij}(s_i)|$ by $\sum_{j=1}^{p} \lambda_j a_{ij}(s_i)$. In certain subsequent integrations we have the cofactors of these elements as multipliers in the integrands. By replacing λ_k by its appropriate values $(\nu_i - \sum_{j=1, j \neq k}^{p} \lambda_j A_{ij})/A_{ik}$, $(i = 1, 2, \cdots, p; k = 2, 3, \cdots, p)$, in the several integrands of the exponential indices, we may evaluate the limits of the parts which do not contribute zero as the so-called mean value of f.

32. Professor A. S. Besicovitch: *Fundamental geometric properties of linearly measurable plane sets of points.*

The notion of a linearly measurable plane set of points was introduced in 1914 by Carathéodory. These sets (A) include as a special case rectifiable plane curves. The present paper develops a general theory of such sets. The notion of the density and of the tangent are introduced and discussed. The sets (A) are classified as regular and irregular. The regular sets (A) in many respects are analogous to the rectifiable curves. For instance, the tangent exists at almost all points of a regular set (A); to any regular set (A) of measure L and any positive number ϵ, there corresponds a set (finite or denumerable) of rectifiable curves, of total measure $< L + \epsilon$, which contains almost all the points of (A). On the other hand, no such properties exist for the irregular sets (A). They are entirely dissimilar to the rectifiable curves, which is shown by many examples. In one of these, an irregular set (A) is constructed, of measure > 0, whose projection on any straight line has measure 0.

33. Professor Harold Hotelling: *Correlations with incomplete data.*

The data for a correlation problem consist essentially of a matrix of observations. One row represents each variate, and one column each instance. Ordinary, partial, and multiple correlation coefficients are all

invariant under certain operations on the matrix. Frequently incompleteness of data leaves certain elements of the matrix unknown. In such cases statisticians have faced the dilemma of discarding every column containing an unknown element, which may so far reduce the number of instances as to rob the results of significance, or of using all possible instances for calculating each ordinary correlation, which may give impossible values for the multiple and partial correlations. But regarding the data as obtained by a random sampling of a normal distribution, it is possible to escape both horns of the dilemma in determining the most likely values, which are necessarily consistent, in the aggregate sampled. The equations for doing this are presented. By similar means an explicit solution is obtained for the following problem. Two variables are known a priori to be uncorrelated, but in a sample there is usually some correlation. What corrections should on this account be applied to the observed correlations of these variables with others, and to partial and multiple correlations involving them?

34. Professor Harold Hotelling: *Spaces of statistics and their metrization.*

An ordinary statistical graph is a geometric figure whose essential properties are invariant only under a group of transformations even more limited than that of euclidean geometry. However, certain statistical problems admit of geometric representation in a manner for which the concepts of differential geometry have significance. One way of metrizing the space of statistical variables is by means of the quadratic form appearing in the normal law of error. This becomes a differential form in various problems in evolution and in economics. Examples of the latter are given in a paper by the author in the September, 1927, number of the Journal of the American Statistical Association.

35. Professor A. J. Kempner: *On the shape of polynomial curves.* (Second paper.)

This is a continuation of a paper presented to the Society in 1919 under the same title. In the earlier paper, the problem treated was to prove the existence of polynomial curves $y=a_0x^n+\cdots+a_n$ (all quantities involved real) with a prescribed arrangement (arbitrary except for obvious restrictions) of the order of magnitude in which the successive maxima and minima are arranged. In that paper, points of inflection were not assumed on the curves. In the present paper, by considering them as being generated by letting successive maxima and minima coincide, with subsequent further continuous variation of the coefficients, points of inflection are taken into account.

36. Professor H. E. Buchanan: *On the oscillations of three finite masses near the equilateral triangle solutions.*

In this paper the author discusses the equations of variation near the equilateral triangle positions. The characteristic exponents are found, and

their nature depends on a certain relation between the masses. In case one of the masses is infinitesimal, these exponents reduce to those given by Moulton in his celestial mechanics. The instability of the equilateral triangle positions is proved, and two types of periodic orbits are shown to exist.

37. Professor H. E. Buchanan: *Note on a certain memoir of Liouville's.*

In a very interesting paper published in Connaissance des Temps, 1845, Liouville discusses the stability of the straight-line solutions of the three-body problem. His result depends on the solution of the equation

$$\alpha^4+(2n'^2-n^2)\alpha^2+(n'^2-n^2)(2n^2+n'^2)=0.$$

He shows that n'^2-n^2 is negative for all values of the masses, and consequently the straight line solutions are unstable because of the presence of a term of the type $e^{\alpha t}$, α real and positive. In the present note it is shown that Liouville's equation is exactly the same as an equation of Buchanan's in a paper in the American Journal of Mathematics (vol. 45 (1923)), p. 98, equation 12), where the periodic orbits near the straight-line solution are discussed, except that Liouville always keeps his three bodies in a straight line. It follows that Liouville's equation is the same as Moulton's *Periodic Orbits*, Carnegie Institution Publication No. 161, and Plummer's Monthly Notices, Royal Astronomical Society, vol. 62 (1901), if one of the masses is infinitesimal.

38. Professor L. M. Graves: *Discontinuous solutions in space problems of the calculus of variations.*

The extension of the Jacobi condition to the case of minimizing curves with corners, the so-called discontinuous solutions, was given by Carathéodory for problems in the plane. His condition is apparently not extensible to space problems, and the details of the proof are very complicated for the plane case. The new form of the condition of Carathéodory obtained by the present author states that the functional determinant of the family of "extremaloids" or broken extremals, whose zeros correspond to points of contact with the envelope of the family, shall not change sign at a corner of a minimizing curve. The statement and proof are practically as simple for n-dimensional problems as for the plane problem. When modified in the usual way, the new condition fits into the familiar set of sufficient conditions for a minimum.

39. Professor D. C. Gillespie: *On infinite determinants and their associated systems of linear equations.*

If in a normal determinant each element of the principal diagonal be diminished by unity, the resulting double array of elements are the terms of an absolutely convergent double series. The normal determinant converges; if its value is different from zero and the constant terms in the

associated system of equations form a bounded set, the equations have one and only one bounded solution. The striking fact that the solution obtained is the only bounded solution is, of course, due to the assumption made concerning the terms of the principal diagonal. The determinants considered in this paper have in common with normal determinants the property that the elements of the principal diagonal approach unity, but differ from them in that the absolute value of the elements in diagonals immediately below the principal diagonal are allowed to approach unity. The determinants are shown to converge, and a solution is obtained for the system of equations associated with each non-vanishing determinant, the solution being the only bounded one if a bounded solution exists. If no bounded solution exists, the solution found is the only solution in which the nth unknown is less in absolute value than the nth term of a certain sequence.

40. Professor G. Y. Rainich: *Deviation from normality of congruences of curves.*

For a congruence of "equidistant" curves in curved n-dimensional space certain expressions which vanish in the case of a normal congruence are introduced as giving in the general case a measure of deviation from normality in the neighborhood of a curve. Integrals of these expressions (which represent an antisymmetric tensor) over a surface cutting the curves of the congruence are introduced, as giving the *total deviation* for the corresponding portion of the congruence. Conditions are considered (for $n>3$) under which this total deviation vanishes over a closed surface. In the case for which $n=5$, the Kaluza-Klein interpretation of electromagnetism is obtained. The centrosymmetric case is worked out as an example of the general theory.

41. Professor D. V. Widder: *On the expansion of analytic functions of the complex variable in generalized Taylor's series.*

In a previous paper the representation of a real function in generalized Taylor's series was discussed. The methods there employed are not applicable to the complex case. In the present paper new methods are developed and the representation of an arbitrary analytic function is obtained. The result is comparable with a result of Birkhoff's given in the Comptes Rendus, 1917.

42. Mr. J. J. Gergen and Professor D. V. Widder: *On Taylor's series admitting the circle of convergence as a singular curve.*

This paper obtains a new criterion that the circle of convergence of a power series be a singular curve. The criterion is a generalization of one recently given by S. Mandelbrojt. By an application of the result, an elementary proof of a familiar theorem on lacunary series is given.

43. Professor R. L. Moore: *A separation theorem.*

In this paper the following theorems are proved. (I) If, in a plane S, M is a bounded continuous curve, K is a closed subset of M, and H is a subset of K and of some connected subset of $K+(S-M)$ but no point set of which H is a proper subset satisfies these conditions, then if the continuum E is a subset of $K-H$, there exists a simple closed curve lying wholly in $M-K$ and separating H from E. (II) If, in a plane S, K is a closed subset of a continuous curve M, and A and B are two points of M, then if A and B lie in a connected subset of $K+(S-M)$ they lie in a *closed* and connected subset of $K+(S-M)$. (III) In order that a bounded continuum M in a plane S should be a continuous curve, it is necessary and sufficient that if t is an arc and H is an open subset of $M+t$, and A and B are points of $S-H$ which are *weakly* separated by H, then H contains a simple closed curve that separates A from B. The first of these theorems may be regarded as an extension of a theorem of Zoretti's.

44. Mr. J. L. Dorroh: *Concerning a set of metrical hypotheses for geometry.*

In his paper *Sets of metrical hypotheses for geometry* (Transactions of this Society, vol. 9 (1908), pp. 487–512) R. L. Moore raises the question whether it follows from the set O of order axioms and the set C of congruence axioms employed therein that every segment has a mid-point. In the present paper this question is answered in the affirmative.

45. Dr. G. T. Whyburn: *On a problem of W. L. Ayres.*

The author shows that if a plane continuous curve M does not contain more than a finite number of simple closed curves of diameter greater than any preassigned positive number, then every connected subset of M is arcwise connected. This answers a question raised by W. L. Ayres in his paper *Concerning continuous curves of certain types.* (See abstract in this Bulletin, vol. 32 (1926), p. 307; the paper will appear in Fundamenta Mathematicae.) In proving this proposition it was shown that every continuous curve satisfying the hypothesis of this theorem has the property that each of its maximal cyclic curves contains only a finite number of simple closed curves. Then with the aid of the author's theorem that every connected subset of a continuous curve is arcwise connected provided that every connected subset of each of its maximal cyclic curves is arcwise connected (see my paper *Concerning the structure of a continuous curve,* presented to the Society Dec. 31, 1926; offered to the American Journal), the proof is easily completed.

46. Dr. G. T. Whyburn: *Concerning the cut points of continua.*

If X is any point set, let X^* denote a subset of X obtained by omitting from X some countable set of points. Let M be any plane continuum, and G the set of all the cut points of M. Then (1) if H is any uncountable subset

of G, there exist two points of H which are separated in M by uncountably many points of H; (2) there exists a set G^* (possibly null) such that for each point X of G^*, $M-X$ is the sum of two connected point sets; (3) every collection of mutually exclusive subcontinua of M each of which contains at least one point of G is countable; (4) for every subset E of G there exists an E^* no point of which belongs to any subcontinuum of M containing no other point of E; (5) there exists a G^* every point X of which is an end point of every continuum obtained by adding X to some maximal connected subset of $M-X$; (6) there exists a G^* such that M is connected im kleinen at every point of G^*; (7) there exists a G^* such that every subcontinuum N of M is connected im kleinen at every point of $G^* \cdot N$; (8) there exists a G^* every point of which is a point of Menger order two of M.

47. Dr. G. T. Whyburn: *Concerning certain kinds of continuous curves and other continua.*

The following theorems are proved. (1) Every connected and relatively open subset of an arcwise connected im kleinen point set is arcwise connected. (2) If H denotes the sum of the boundaries of all the complementary domains of a continuous curve M every subcontinuum of which is a continuous curve, then (a) H is arcwise connected, and (b) $M-H$ is totally disconnected. (3) If every maximal cyclic curve of a continuous curve M is a simple closed curve, then M is a Menger "regular curve" and, indeed, for any $\epsilon>0$, M is the sum of a finite number of continua each of diameter $<\epsilon$ and no two of which have more than one point in common. The case where M is the boundary of a domain is a special case of (3). (4) Every "regular point" (Menger) of a continuum M is regularly accessible from every complementary domain of M to the boundary of which it belongs. (5) Every cut point of a continuum M which is of Menger order two belongs to the boundary of only one complementary domain of M. (6) If every point of a continuum M which contains no domain is *regularly accessible from* $S-M$, then M is a Menger regular curve.

48. Dr. W. L. Ayres: *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve.*

A subset N of a plane continuous curve M is said to be connected through N and $S-M$ if there exists a subset H of $S-M$ such that $N+H$ is connected. (1) If K is a bounded subcontinuum of M, T is a totally disconnected cutting of K, and x and y are points of $K-T$ which cannot be connected through $K-T$ and $S-M$, then there exists a simple closed curve J which is a subset of M, separates x and y, and is such that $J \cdot K$ is a subset of T. (2) A bounded M-domain is simply connected with respect to M if and only if its M-boundary is connected through B and $S-M$. (3) If K is a closed cutting of M such that if H is any component of K then $K-H$ is not a cutting of M, then K is connected through K and $S-M$. (4) In order that a plane continuum K be a continuous curve it is sufficient

that if A is any arc, H is a closed subset of $K+A$, and x and y are two points of H which can be connected through H and $S-K$, then x and y can be connected through H and $S-K-A$ by a continuum. (R. L. Moore has shown the necessity of (4).)

49. Professor W. A. Wilson: *On irreducible cuts of the plane between two points.*

If F is an irreducible cut of the plane between two points, any component of the complement of F which has F as its frontier is called a *principal* component. A decomposition of a continuum C into two subcontinua is called *irreducible* if C is not the union of one of the subcontinua and a proper subcontinuum of the other. The following theorem is obtained. Let F be a bounded irreducible cut between two points which is decomposable and let $F=H+K$ be an irreducible decomposition of F into proper subcontinua. According as the number of principal components of the complement of F is finite or infinite, $H \cdot K$ is the sum of the same number or an infinite number of closed subsets, no pair of which have common points, and H and K are both irreducible between each pair of these sets. This is an analogue of C. Kuratowski's theorem regarding a completely irreducible cut of the plane, i.e., a cut of the plane which is the frontier of *every* component of its complement.

50. Dr. C. H. Langford: *An analysis of some general propositions.*

This paper appears in full in the present number of this Bulletin.

51. Professor J. L. Walsh: *On the approximation to harmonic functions by harmonic polynomials.*

This paper has appeared in the September-October number of this Bulletin.

52. Professor J. L. Walsh: *On the expansion of analytic functions in series of polynomials and of other analytic functions.*

This paper presents some extensions of the author's previous work on expansions of arbitrary functions in terms of polynomials (Transactions of this Society, vol. 26 (1924), pp. 155–170). In particular, there are considered the expansion of a discontinuous function, the analog of Gibbs's phenomenon, the analog of Abel's theorem and its converse, and equivalence of expansions not merely on a single curve but on a one-parameter family of curves.

53. Dr. G. M. Merriman: *On sufficient conditions for the convergence and Cesàro summability of the allied series of a double Fourier series.*

This paper has appeared in the September-October number of this Bulletin.

54. Miss Florence M. Mears: *Riesz summability for double series.*

This paper defines for double series a mean, $(R:\lambda, p; \mu, r)$, corresponding to the Riesz definition, (R, λ, p), for summing simple series. It includes theorems relative to the regularity and total regularity of the extended definition; it establishes a relation between methods of summation of the same type, (λ, μ), when either p or r, or both p and r, are changed; a relation between methods of summation of the same order, (p, r), when either λ or μ, or both λ and μ, are changed; certain necessary conditions for the Riesz summability of double series; theorems for the Dirichlet and Cauchy products of double series corresponding to those of Mertens, Cauchy and Abel for the Cauchy product of simple series; a sufficient condition for the summability of the product of two double series to the correct sum.

55. Dr. W. M. Whyburn: *Existence and oscillation theorems for non-linear differential systems of the second order.*

The method of successive approximations is used to establish a general existence theorem for the system of first-order differential equations $dy_i/dx = \sum_{j=1}^{n} A_{ij}(x, y_1, \cdots, y_n)y_j$, $(i = 1, 2, \cdots, n)$. The second-order system $dy/dx = K(x, y, z; \lambda)z$, $dz/dx = G(x, y, z; \lambda)y$, is considered along with boundary conditions of the type $y(a, \lambda) = y(b, \lambda) = 0$. Existence and oscillation theorems are established for cases where K and G satisfy conditions similar to those imposed in treating the ordinary linear systems of the Sturmian type. A treatment of linear Sturmian systems is given in which several of the usual conditions on the coefficients are replaced by symmetric conditions on K and G. The methods of this paper are based on transformations of the type $y = u \cos v$, $z = u \sin v$.

56. Professor W. P. Webber: *On a generalization of the Weierstrassian theory of elliptic functions.*

Defining the variable u by the vector equation $u = i_1x_1 + i_2x_2 + \cdots + i_px_p$, where $i_1, i_2, \cdots, i_p$ are unit vectors in the directions of a system of mutually perpendicular axes $OX_1, OX_2, \cdots, OX_p$, and $x_1, x_2, \cdots, x_p$ are real coordinates of a point referred to these axes; defining w by the vector equation $w = 2i_1m_1\omega_1 + 2i_2m_2\omega_2 + \cdots + 2i_pm_p\omega_p$, where $m_1, m_2, \cdots, m_p$ take all integer values independently and where $\omega_1, \omega_2, \cdots, \omega_p$ are arbitrary real constants; and employing a generalization of ordinary vector algebra, the author constructs the function

$$F_p(u) = (1/u_p) + \sum{}'[1/(u-w)^p - 1/w^p],$$

which is valid and possesses p periods $2\omega_1, 2\omega_2, \cdots, 2\omega_p$; the accent on $\sum$ indicates that $w = 0$ is omitted. Convergent series valid in the fundamental cell are obtained for the cases $p = 3$ and $p = 4$. Functions analogous to the classical functions σ, ζ, Z are given.

57. Professor G. D. Birkhoff: *Reversibility in dynamics.*

In this paper it is proved that a necessary and sufficient condition for complete stability of a periodic motion is the reversibility of the time t

in the differential equations, provided that the dependent variables are also subject to transformation.

58. Professor G. D. Birkhoff: *The periodic motions near a stable periodic motion of a dynamical system.*

On the basis of a very simple lemma concerning the invariant points of a transformation of the plane in the neighborhood of a given invariant point, the author shows that there exist infinitely many periodic motions in an arbitrarily small neighborhood of a stable periodic motion of a dynamical system with two degrees of freedom. This lemma is closely related to the extended form of Poincaré's last geometric theorem.

59. Professor G. C. Evans: *Note on a generalization of a theorem of Bôcher.*

Let $u(M)$ be a "potential function of its vector derivative or gradient" and satisfy Bôcher's equation $\int_s D_n u ds = 0$ "almost everywhere" on rectangles in the plane open region Σ. Then $u(M)$ has at most removable discontinuities in Σ, and when these are removed by changing the values of $u(M)$ at most in a set of measure zero, it becomes continuous with all its derivatives and satisfies Laplace's equation at every point of Σ. The theorem remains valid if the set of rectangles is replaced by any "translatable regulat net"; and a corresponding theorem holds in any number of dimensions. Since it is not assumed that the curves on which Bôcher's equation holds are level curves of a harmonic function, an indirect proof is necessary.

60. Professor G. C. Evans: *Poisson's equation in the plane and other equations of elliptic type.*

The equation considered is $\int_s D_n u \, ds = \Phi(s)$, where $\Phi(s)$ is the function of curves with regular discontinuities which corresponds to a given arbitrary completely additive function of point sets $\Phi(e)$, in an open finitely connected region Σ. The principal solution is a "potential function of its vector derivative or gradient," belongs to the class (ii) of the author's book on *Logarithmic Potential*, and takes on zero boundary values almost everywhere "in the narrow sense." Both types of the first boundary value problem are uniquely solvable in their respective classes of functions. The corresponding problems for the sphere may also be solved. By means of integral equations, other equations of elliptic type may be treated, as in the Cambridge Colloquium. The given equation more or less reduces to Poisson's equation if $\Phi(s)$ or $\Phi(e)$ is assumed to be absolutely continuous.

61. Dr. C. F. Roos: *A mathematical theory of depreciation and replacement.*

In as much as a machine is usually replaced by another machine before its useful life has ended, it is quite important to consider the problem of determining the time at which a machine in operation should be replaced by another machine whose operating expense is different so that a maximum profit is obtained for some period of time extending from an initial time t_1

through a replacement time ω to some final time t_2 greater than ω. When formulated as a problem in the calculus of variations, this replacement problem is a type of Lagrange problem with variable end points and discontinuous integrand. In this paper a general formulation is made of a replacement problem and a special example of an actual economic situation is solved in some detail.

62. Dr. C. F. Roos: *The problem of depreciation in the calculus of variations.*

This paper will appear in full in an early number of this Bulletin.

63. Professor W. J. Trjitzinsky: *Expansion in series of non-inverted factorials.*

Using Cauchy's result $f(z)=[1/(2\pi i)]\int_C[f(t)/(z-t)]dt$ and $\int_0^1 u^n(1-u)^v du=(n!)/[(v+1)\cdots(v+n+1)]$, we expand $1/(z-t)$ in a series of direct factorials in z uniformly convergent when $R(t)<-1$. It follows that if $f(z)$ is analytic on and outside a closed contour C situated to the left of $R(t)=-1$, then for $R(z)>0$,

$$f(z)=a_0+a_1z+a_2z(z-1)+\cdots+a_nz(z-1)\cdots(z-n+1)+\cdots,$$

where $a_n=-[1/(2\pi i)]\int_C[f(t)/(t(t-1)\cdots(t-n))]dt$, and $|a_n|<h/(n!)$. This can be generalized for any position of C.

64. Professor W. J. Trjitzinsky: *A class of infinite products and an application to the theory of the gamma function.*

We consider functions

$$La_i(z)=\frac{e^{-\alpha z}}{z}\prod_{n=1}^{\infty}e^{z/a_n}\left(1+\frac{z}{a_n}\right)^{-1},$$

where α is analogous to the Eulerian constant, and the a_n are under certain restrictions; these functions reduce to the gamma function when $a_n=n$. A multiplication theorem for products $La_i(z)La_i(z+p_1)\cdots La_i(z+p_n)$ is obtained, where the p_i satisfy certain inequalities. When $a_i=i$, this theorem yields a multiplication theorem for the gamma function that is disctinct from the classical theorem.

65. Dr. W. L. Ayres: *Concerning continuous curves in n dimensions.*

In this paper the following theorems which have been proved previously for two dimensions are proved for n dimensions: (1) in order that a point P of a continuous curve M be an end point of M it is necessary and sufficient that P be a non-cut point which lies on no simple closed curve of M; (2) in order that a continuous curve be cyclically connected, it is necessary and sufficient that it contain no cut point; (3) in order that there exist a point which separates the points A and B in the continuous curve M it is necessary and sufficient that every two arcs of M whose end points are A and B have an interior point of both in common.

66. Professor L. E. Dickson: *Complete solution of the Waring-Kamke problem on sums of integral values of quadratic functions.*

Let $q(x)$ be an integer $\geqq 0$ for every integer $x \geqq 0$. For every $s \geqq 5$, there is found the least integer e_s such that every positive integer is a sum of s values of $q(x)$ and e_s numbers 0 or 1. There is deduced the least value L of $s+e_s$ for $s \geqq 5$. Next, it is shown that $4+e_4 \geqq L$ except for specified special cases, and then $4+e_4 = L-1$. Except for these special cases, it is not necessary to carry out the very difficult work of finding the exact value of e_4. By the new method developed, it is relatively simple to find each e_s for $s \geqq 5$. The memoir giving the complete solution of this problem will appear in the American Journal.

67. Professor J. V. Uspensky: *On the number of representations of integers by certain ternary quadratic forms.*

Denoting by (a, b, c) the form $ax^2+by^2+cz^2$, the author gives the expression for the number of representations of integers by the following forms: $(a, b, c) = (1, 2, 3)$, $(1, 2, 6)$, $(2, 3, 6)$, $(1, 3, 6)$, $(1, 1, 3)$, $(1, 3, 3)$, $(1, 1, 6)$, $(1, 6, 6)$, $(1, 1, 5)$, $(1, 5, 5)$, $(1, 2, 5)$, $(1, 5, 10)$, $(1, 2, 10)$, $(2, 5, 10)$.

68. Professor J. V. Uspensky: *On the connections between numbers of representations of integers by two totally different quadratic forms.*

For certain couples of quadratic forms involving three and four variables without any apparent relationship to each other, there exist unexpected relations between the numbers of representations of integers by these forms. The author gives several instances of this rather remarkable phenomenon.

69. Professor Arnold Emch: *On the mapping of the sextuples of the symmetric substitution group G_6 in a plane upon a quadric.*

This paper appears in full in the present number of this Bulletin.

R. G. D. RICHARDSON, *Secretary.*

THE SECOND MADISON COLLOQUIUM

The eleventh colloquium of the American Mathematical Society and the second Madison Colloquium was held at Madison, Wis., in conjunction with the thirty-third summer meeting of the Society from September 6-10, 1927, the lecturers being Professor E. T. Bell, California Institute of Technology, and Professor Anna Pell Wheeler, Bryn Mawr College. The first Madison Colloquium was held in 1913, the lecturers being Professors L. E. Dickson and W. F. Osgood.* In opening the colloquium, President Snyder made the observation that this was the first colloquium of the Society at which a woman had been invited to give a course of lectures. Also that geographically the East and West were represented in the speakers, the Middle West in the place of meeting, and the South in a recent addition to the volumes of the colloquium series.

The lectures were divided as is customary into two groups of five lectures each, and were given from Tuesday to Saturday in the auditorium of Sterling Hall. One hundred twenty-seven persons attended these lectures, the largest number registered for any colloquium so far held, though in comparison with the attendance at Ithaca in 1924 (122) the gradient seems to be on the decrease. The names of those attending the lectures follows.

L. K. Adkins, F. E. Allen, Altshiller-Court, W. E. Anderson, Archibald, R. W. Babcock, Barnett, L. Battig, E. R. Beckwith, E. T. Bell, Birkhoff, Blichfeldt, Bray, Brenke, Brink, E. W. Brown, H. E. Buchanan, L. H. Bunyan, Bussey, Cairns, C. C. Camp, Carlson, G. N. Carmichael, R. D. Carmichael, Chang, Chittenden, J. T. Colpitts, Conkwright, C. F. Craig, H. V. Craig, Crathorne, Curry, Dalaker, H. T. Davis, Dean, Dickson, Dines, Dowling, Dresden, J. M. Earl, G. C. Evans, H. P. Evans, Feinler, W. B. Ford, Fry, Gaba, Gergen, Gill, D. C. Gillespie, W. C. Graustein, M. C. Graustein, L. M. Graves, Griffiths, V. G. Grove, Guggenbühl, Harkin, E. Hart, W. L. Hart, Hartung, E. R. Hedrick, Hildebrandt,

* A resumé of the first ten colloquia of the Society was given in connection with the report of the second Ithaca Colloquium, this Bulletin, vol. 32 (1926), pp. 25–27.

Hoersch, Hollcroft, F. Hopkins, Hosford, Hotelling, J. C. Hughes, L. Ingold, Ingraham, D. Jackson, Jensen, Kearney, Kempner, Krathwohl, E. P. Lane, Langer, Latimer, Logsdon, J. V. McKelvey, March, Maria, A. S. Merrill, N. H. Mewaldt, Michal, Mickelson, Miser, E. J. Moulton, Olson, F. W. Owens, H. B. Owens, Palmer, Parkinson, Pettit, Pierce, R. G. D. Richardson, H. L. Rietz, W. C. Risselman, Roever, Roos, Roth, Runge, R. G. Sanger, Shewhart, Shohat, Sinclair, Skinner, Slaught, Slichter, V. Snyder, W. A. Spencer, Stouffer, Swartzel, Tate, J. H. Taylor, E. M. Thomas, Torrey, Trjitzinsky, J. V. Uspensky, Van Vleck, Wall, W. Weaver, Westfall, A. P. Wheeler, Widder, Worthington, Yanney, Yeaton.

Below are given the synopses of the lectures as prepared by the lecturers in the order in which they were given:

ALGEBRAIC ARITHMETIC

By Professor E. T. Bell

I. *Definition and scope of algebraic arithmetic.* Arithmetical theories as opposed to algebraic or analytic. The algebraic varieties useful in arithmetic; irregular fields, rings, rays, semigroups; the algebras of infinite vectors.

II. *The algebra of unique factorization.* Applications to an arithmetic of functions defined through any unique factorization law.

III. *The algebra of parity.* Definition of parity functions. Division of parity and equality of functions.

IV. *Algebraic arithmetic of multiply periodic functions.* Arithmetical identities between arbitrary functions having parity. Applications to theorems on representations of integers in higher forms. Applications to quadratic forms and class number relations. Singly infinite class number relations.

V. *Arithmetical structure.* Arithmetization of an arithmetizable theory. Solution in terms of symbolic logic of certain of the postulate systems of rational arithmetic.

THE THEORY OF QUADRATIC FORMS IN INFINITELY MANY VARIABLES AND APPLICATIONS

By Professor Anna Pell Wheeler

I. *Spaces of infinitely many dimensions.* Vectors. Fundamental quadratic form. Scalar product. Orthogonality.

Construction of coordinate systems corresponding to a given fundamental form. Properties of coordinate systems in euclidean spaces. Properties of coordinate systems in non-euclidean spaces. Connection between spaces of different types and classes of functions. General coordinate systems.

II. *Infinite matrices and linear transformations.* Matrices, bilinear forms, and linear transformations. Limitedness with respect to Hilbert space. Limitedness with respect to other spaces. Complete continuity in the Hilbert sense. Extension. Infinite determinants. Some sufficient conditions for the alternative theorems. The Fredholm determinant. Principal solutions of homogeneous equations.

III. *Reduction of quadratic and bilinear forms.* Nature of invariants of linear transformations in the symmetric case. A necessary condition for the existence of invariants in the symmetric case. Carleman's sufficient condition and extension in the symmetric case. Symmetrizable cases. Semi-symmetrizable cases. Simultaneous reduction of two quadratic forms. Solutions of more general linear equations.

IV. *Application to the theory of linear functional equations.* Integral equations with continuous symmetric kernels. Integral equations with singular symmetric kernels. Symmetrizable kernels. Semi-symmetrizable kernels. Integro-differential equations. Systems of integral equations. Linear functional equations of various types.

V. *Application to the theory of linear differential equations.* Ordinary linear differential equations, orthogonal and polar. Singular linear differential equations. Elliptic differential equations. Hyperbolic differential equations. The Bliss system of differential equations.

Both sets of lectures will be published in book form by the Society. Professor Bell's will appear in the fall of 1927, and Professor Wheeler's probably in 1928.

T. H. HILDEBRANDT

AN ANALYSIS OF SOME GENERAL PROPOSITIONS*

BY C. H. LANGFORD

1. *Introduction.* General propositions are commonly described as being those propositions which arise from matrices by generalization, that is, as being such propositions as can be derived from some matrix by the attachment of an applicative, "some" or "every," to each variable constituent of the matrix.† In this paper an analysis of general propositions is suggested, which results from a slightly different view of the relations of general propositions to matrices. It appears that the analysis which is suggested involves a generalization of the ordinary analysis.

2. *Unanalysed Propositions.* We may begin with a consideration of elementary matrices whose values are elementary functions of unanalysed propositions, such as, for example, $p \supset q$. Let t_0 denote any elementary proposition. Then we can write, for example, $(t_0).t_0 \vee \sim t_0$, that is, every elementary proposition is true or false. If p_0, q_0 denote elementary propositions, then, of course, $p_0 \supset q_0$ denotes elementary propositions; but this latter, more complex function denotes a narrower range of propositions than does t_0, since whatever is an elementary proposition of the form $p_0 \supset q_0$ must be an elementary proposition of the form t_0, but not conversely. Let t_1 denote any elementary matrix. Then t_1 denotes what denotes elementary propositions; t_1 denotes t_0, $p_0 \supset q_0$, and the like, which denote elementary propositions. Now we can form such functions as $p_1 \supset q_1$, which denote a narrower range of matrices than does t_1.

Since elementary matrices are neither true nor false, we cannot say $(t_1).t_1 \vee \sim t_1$; but we can say

$$(t_1):(t_0).t_0 \vee \sim t_0,$$

* Presented to the Society, September 9, 1927.

† See *Principia Mathematica*, second edition, vol. 1, p. xxiii.

that is, every value of every matrix is true or false. Variables such as t_0, t_1 fall into a hierarchy of values and values of values. We can write such propositions as $(t_1):(\exists t_0).t_0$, that is, some value of every matrix is true (which is false), and $(\exists t_1):(t_0).t_0$, that is, all values of some matrix are true (which is true); and we have, of course,

$$(t_1)(t_0).t_0.\vee.(\exists t_1)(\exists t_0).\sim t_0,$$
$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

and the like. Consider the proposition

$$(t_1).(t_0)t_0\vee(\exists t_0)\sim t_0,$$

that is, every matrix is such that all of its values are true or at least one of its values is false. This proposition, although a consequence of the proposition $(p).p\ \vee\sim p$, cannot be stated without the use of some variable such as t_1. We have also, of course,

$$(t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0,$$

which is different from

$$(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0\ ;$$

and it would seem that the proposition

$$(\exists t_2):(t_1)(\exists t_0).t_0.\vee.(\exists t_1)(t_0).\sim t_0$$

is not equivalent to any proposition which can be stated without the use of some variable such as t_2.

3. *Values of Functions and Species of Functions.* There are two ways in which inferences can be drawn from universal propositions of the kind with which we are concerned: we can replace a universally quantified function by one of the values which it denotes, and we can replace a universally quantified function, as genus, by a more determinate function, as species.* Thus,

$$(t_1).(\exists t_0)t_0\vee(t_0)\sim t_0$$

* See *Principia Mathematica*, second edition, vol. I, p. xxiii.

implies

$$\exists(p_0 \vee q_0) . p_0 \vee q_0 . \vee . (p_0 \vee q_0) . \sim p_0 . \sim q_0,$$

that is,

$$(\exists p_0, q_0) . p_0 \vee q_0 . \vee . (p_0, q_0) . \sim p_0 . \sim q_0.$$

Again,

$$(t_1)(\exists t_0) . t_0 \vee \sim t_0$$

implies

$$(p_1 \vee q_1)\exists(p_0 \vee q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0,$$

that is,

$$(p_1, q_1)(\exists p_0, q_0) : p_0 \vee q_0 . \supset . p_0 \vee q_0.$$

It is to be noted that *genus* and *species* are to be understood in intension: being a function of the form $p_1 \vee q_1$ entails being a function of the form t_1; and this is the ground of the implication. There are two ways in which particular propositions can be inferred: we can infer a particular proposition from an instance of the function which is quantified particularly, and we can infer a particular proposition from a more determinate particular proposition. Thus, $(t_0) . t_0 \vee \sim t_0$ implies $(\exists t_1)(t_0) . t_0$; and $\exists(p_0 . q_0) . p_0 . q_0$ implies $(\exists t_0) . t_0$.

4. *Analysed Propositions.* Heretofore we have been concerned with matrices whose values are elementary functions of unanalysed propositions, and we have illustrated the way in which a hierarchy of functions can be formed in this connection. Thus we may have

$$p_2 \vee q_2,$$

which has as a typical value

$$p_1' . p_1'' . \vee . q_1' . q_1'',$$

which has as a typical value

$$r_0' \supset s_0' . r_0'' \supset s_0'' . \vee . t_0' \supset u_0' . t_0'' \supset u_0'',$$

which has as a value any elementary proposition of this form.* I wish to explain how a similar hierarchy can be formed in connection with analysed propositions. A propo-

* See *Principia Mathematica*, loc. cit., p. xxxi, etc.

sition is a value of a function if the function can be obtained by replacing constituents of the proposition by appropriate variables. We are to consider propositions which are values of relational functions, such as the proposition "*a* gives *b* to *c*." This proposition is a value of each of the functions "*a* gives *b* to *x*," $f(a, b, c)$, $f(a)$, $f(x)$, $f(x, y, z)$, among others. Commonly, one and the same proposition is open to various analyses. If we consider an appropriate constituent of a relational proposition, two entities appear which can be replaced by variables, namely this constituent and the remainder of the proposition. It is, however, often difficult to decide what are the constituents of a proposition which can be replaced by variables.

Now a proposition of the form Rab is a value of the function Ray; and it is clear that Ray can be obtained from Rxy by replacing x by a. We may indicate this order of denoting by writing Rx_1y_0; so that Rx_1y_0 denotes Ray_0, which denotes Rab. Subscripts indicate parameters of various orders, and accordingly they indicate the order of denoting; so that the subscript which attaches to a variable determines the point in the hierarchy at which the variable takes values. Thus a function $f(x, y, z)$ is ambiguous as regards the parametric order of its variable constituents. We may have $f_1(x_0, y_0, z_0)$, which denotes "x_0 gives y_0 to z_0," which denotes "*a* gives *b* to *c*"; or, we may have $f_0(x_1, y_1, z_1)$, which denotes $f_0(a, b, c)$, which denotes "*a* gives *b* to *c*"; and so on. It is clear that, as a result of this analysis, we can attach an applicative, "some" or "every," to the entire function at each stage of the hierarchy. If we apply "some" to $f_1(x_0, y_0, z_0)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_1(x_0, y_0, z_0) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists f) : (x, y, z) . f(x, y, z)$ in the ordinary analysis; whereas, if we apply "some" to $f_0(x_1, y_1, z_1)$ and "every" to $f_0(x_0, y_0, z_0)$, we have

$$\exists f_0(x_1, y_1, z_1) : (f_0(x_0, y_0, z_0)) . f_0(x_0, y_0, z_0),$$

which corresponds to $(\exists x, y, z) : (f) . f(x, y, z)$.

Now although the proposition $(f_1x_0):(f_0x_0).f_0x_0$ corresponds to $(f):(x).fx$, there is an important difference between these propositions which I wish to point out. When we say $(f):(x).fx$, it is presupposed that each value of x can be combined with every value of f to form a significant proposition; whereas, no such presupposition as this is involved when we write $(f_1x_0):(f_0x_0).f_0x_0$. If there should be values of f_0, say f', f'', and values of x_0, say x', x'', such that $f'x'$ and $f''x''$, but such that $f'x''$ and $f''x'$ are non-significant, this fact would not render $(f_1x_0):(f_0x_0).f_0x_0$ either false or non-significant. A matrix is to be regarded as taking such values as exhibit the form of the matrix; and the values taken by a variable constituent of a matrix depend upon the matrix in which the variable occurs. It is therefore possible that $(f_1x_0):(f_0x_0).f_0x_0$ should have a wider reference than $(f):(x).fx$.

5. *Analysed and Unanalysed Propositions.* I wish now to consider functions whose values are elementary functions of analysed propositions, and to inquire how, if at all, the denotative hierarchy for elementary functions of unanalysed propositions can be combined with the hierarchy for analysed propositions. We have such propositions as $(f_1x_0)(\exists f_0x_0).f_0x_0$ and $(\exists f_0x_1)(f_0x_0).f_0x_0$. Now it appears that we can extend the range of t_0, t_1, $\cdots$ so as to include such functions as f_0x_0, f_1x_0, $\cdots$. This can be seen to be possible by noting that f_0x_0 is a species of t_0, and that f_1x_0 is a species of t_1. An analysed function will have the order of its constituents of highest parametric order. It is clear that, for example, $(f_0x_0).f_0x_0 \vee \sim f_0x_0$ follows from $(t_0).t_0 \vee \sim t_0$, and that $(f_0x_0)f_0x_0 \vee (\exists f_0x_0)\sim f_0x_0$ follows from $(t_0).t_0 \vee (\exists t_0)\sim t_0$; and also that $(\exists f_0x_0)f_0x_0$ entails $(\exists t_0)t_0$. Consider the proposition

$$(a) \qquad (t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0.$$

If we choose f_1x_0 as a species of t_1, (a) is seen to entail

$$(f).(\exists x)fx \vee (x)\sim fx\ ;$$

but if we choose f_0x_1 as a species of t_1, (a) is seen to entail

$$(x).(\exists f)fx \vee (f)\sim fx.$$

Functions of any number of variables can replace t_1, but, in any case, doubly-quantified propositions result. Of course, we can assign values to t_1, instead of replacing it by species. We can, for example, substitute fx_0 (where f is constant), so that we have

$$(\exists x)fx \vee (x)\sim fx\ ;$$

or, if we substitute f_0a, we have

$$(\exists f)fa \vee (f)\sim fa.$$

This analysis gives us a certain advantage as against the ordinary analysis. In the ordinary analysis, we can write $(p).p \vee \sim p$, and derive, as a species, $(f,\ x).fx \vee \sim fx$; but this is confined to singly-quantified propositions, that is, to propositions involving a single applicative, "some" or "every"; whereas, we are now able to write, say, $(p_n) \cdots (\exists p_0).p_0 \vee \sim p_0$, which is a proposition involving $n+1$ applicatives.

6. *Multiply-Quantified Propositions.* Let $F(t_0)$ denote an elementary function of t_0. Then it is clear that $(t_0).F(t_0)$ implies $(t_1):(t_0).F(t_0)$. Similarly, $(t_1):(t_0).F(t_0)$ implies $(t_2):.(t_1):(t_0).F(t_0)$. Now $(\exists t_0).F(t_0)$ can be obtained from $(t_0).F(t_0)$; and generally, in a function of the form $(t_n) \cdots F(t_0)$, we can turn any universal variable into a particular variable. Accordingly, any proposition on $F(t_0)$, of whatever degree of quantification, can be obtained from $(t_0).F(t_0)$.

In the argument of the last paragraph, we have used $F(t_0)$ to denote an elementary function of t_0; but I should like to point out that it is not at all necessary to employ such expressions as $F(t_0)$. The proposition $(\exists F):(t_0).F(t_0)$, for example, expresses what is expressed by $(\exists t_1):(t_0).t_0$. We can express the proposition

$$(F):(t_0).F(t_0).\supset.(t_1)(t_0).F(t_0)$$

in the following way. Note that $(\exists t_0).t_0$ is equivalent to $(\exists t_1)(\exists t_0).t_0$; so that we can write

$$(t_2):(\exists t_1)(\exists t_0).\sim t_0.\vee.(t_1)(t_0).t_0.$$

Mr. Russell has shown how elementary functions of propositions that are not elementary can be derived from elementary matrices.* Thus, $(x).fx \vee \sim fx$ implies $(x)(\exists y).fx \vee \sim fy$, which is equivalent to $(x)fx \vee (\exists y)\sim fy$. In precisely the same way,

$$(t_1)(t_0).t_0 \vee \sim t_0$$

implies

$$(t_1):(t_0)(\exists t_0').t_0 \vee \sim t_0',$$

which is equivalent to $(t_1).(t_0)t_0 \vee (\exists t_0)\sim t_0$, which implies

$$(t_1)(\exists u_1).(u_0)u_0 \vee (\exists t_0)\sim t_0,$$

which is equivalent to

$$(\exists t_1)(t_0)t_0 \vee (t_1)(\exists t_0)\sim t_0.$$

It is clear that elementary functions of propositions of the first and second orders can be derived in this way, whatever the degree of quantification of these propositions. Moreover, from $(t_1).(\exists t_0)t_0$ we derive both $(\exists x)fx$ and $(\exists f)fa$, although one of these latter propositions would be said to be of the first order and the other of the second order. Furthermore, in a proposition such as $(t_0)t_0$ there seems to be no reason for restricting t_0 to elementary propositions; it can denote propositions of the first and second orders as well. Let "$(\exists x).a$ gives b to x" be denoted by $f(a, b)$. Then

$$(\exists a,b)f(a,b) \vee (a,b)\sim f(a,b)$$

results from

$$(t_1).(\exists t_0)t_0 \vee (t_0)\sim t_0.$$

It does not matter that f is not elementary, since it occurs as an unanalysed constituent.

HARVARD UNIVERSITY

* See *Principia Mathematica*, loc. cit., *8.

ON THE POLYNOMIAL CONVERGENTS OF POWER SERIES*

BY W. M. WHYBURN

In this paper we will consider the power series

$$\sum_{i=0}^{i=\infty} a_i z^i$$

with a unit radius of convergence. M. B. Porter† defined a set of convergents for this series in such a way that the set had many interesting properties.‡ He proved that at least one point of the unit circle was a limit point of the zeros of these convergents, and that in the neighborhood of every point of this circle some of the convergents took on values whose moduli were arbitrarily small. Jentzsch§ has shown that every point of the unit circle is a limit point of the zeros of the complete set $\{f_n(z)\}$ of convergents defined by

$$f_n(z) = \sum_{i=0}^{i=n} a_i z^i$$

for all positive integral values of n. It is the purpose of the present paper to show that the set of convergents defined by Porter has every point of the unit circle as a limit point of its zeros. As a special case of this result we get the theorem of Jentzsch. Other properties of these convergents are developed also.

Let $f(z)$ be the analytic function defined by the above power series inside of the circle $|z| = 1$ and let z_0 be a point

* Presented to the Society, April 15, 1927.

† Annals of Mathematics, vol. 8 (1906–7), pp. 189–192.

‡ Of extreme importance is the fact that Porter embodied in this set of convergents all of the properties that were later assigned by Montel (see his *Leçons sur les Séries de Polynomes etc.*, Paris, 1910) to his *normal families* of functions.

§ Acta Mathematica, vol. 41 (1917), pp. 253–270.

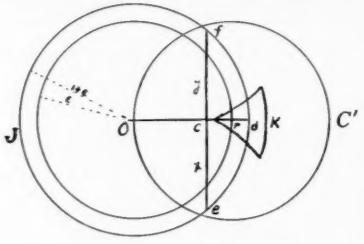
such that $|z_0|>1$. Let $\{P_{n_i}(z)\}$ be the set of convergents defined by

$$P_{n_i}(z) = a_{n_i}z^{n_i}[a_0z^{-n_i}/a_{n_i} + a_1z^{-n_i+1}/a_{n_i} + \cdots + 1],$$

where $|a_{n_i}z_0^{n_i}|$ is greater than any of the expressions $|a_jz_0^j|$, for $j=0, 1, 2, \cdots, n_i-1$.

THEOREM I. *Every point of the circle $|z|=1$ is a limit point of zeros of the polynomials of the set $\{P_{n_i}(z)\}$.*

PROOF. Assume the theorem false; then there exists a point p on the unit circle and a circle C' with its center at p such that there are no zeros of $f(z)$ or $\{P_{n_i}(z)\}$ within or on C'. Draw the chord xy within C' cutting the arc of $|z|=1$



that lies within C' in such a way that the point p is the midpoint of the intercepted sub-arc of this arc. Choose a positive number ϵ so that the circle $J: |z|=1+\epsilon$ cuts this chord (produced) at two points e and f, both of which lie within C'. Porter* has shown that we can pick a sub-sequence $\{P_{n_j}(z)\}$ of $\{P_{n_i}(z)\}$ such that for all n_j greater than a fixed index N_1, $P_{n_j}(z)$ has all of its zeros except a fixed finite number, m', inside of the circle J. By Hurwitz'† theorem we can determine an index N_2 so that for all $n_j>N_2$, the m' zeros of $P_{n_j}(z)$ that are not within J will lie within a set G of circles whose centers are at m' fixed points‡ and whose

* Loc. cit., p. 191.

† Acta Mathematica, vol. 40 (1916), pp. 179–183.

‡ These points are the zeros of the analytic function defined by $\{\overline{P}_{n_j}(z)\}$ outside of J, where $P_{n_i}(z)=a_{n_i}z^{n_i}\overline{|P}_{n_i}(z)|$.

radii are equal to a given positive constant ϵ'. We fix ϵ' so that none of the circles of the set G will have anything in common with the region C, where C lies within C' and is bounded by the arc *edf* and the chord *ecf* (c and d are the points in which the ray Op cuts the chord *ef* and the arc *ef*, respectively). Since $f(c)$ is different from zero, let $|f(c)|=2k$ and since $\lim_{n_j\to\infty} P_{n_j}(c)=f(c)$ uniformly, choose an index N_3 so that for every $n_j>N_3$, $|P_{n_j}(c)|>k$.

We determine* a region K that lies within C', contains p and infinitely many points outside of J, has nothing in common with any of the circles of G, and such that the distance from any point of the interior of J that does not belong to C or the boundary of C to any point of K is as great as the distance from that point to c. Let a positive number m be chosen so that the distance from any point of K to any of the circles of the finite set G is not less than m, and let another positive number M be chosen so that none of the above distances are greater than M.

By Porter's theorem, we can find an index N that is greater than N_1, N_2, N_3, such that $|P_N(s)|<m^{m'}k/(4M^{m'})$, where $P_N(z)$ belongs to $\{P_{n_j}(z)\}$ and s is a point of K. Let z_1, z_2, $\cdots$, $z_{m'}$ be the zeros of $P_N(z)$ that are not within J and let $z_{m'}+1$, $\cdots$, z_N be the remaining zeros of this function;

$$P_N(z) = a_0(z-z_1)(z-z_2)\cdots(z-z_{m'})\cdots(z-z_N).$$

$$(1)\qquad |P_N(s)| = |a_0|\,|(s-z_1)|\cdots|(s-z_{m'})|\,|(s-z_{m'+1})|\cdots|(s-z_N)| \geqq m^{m'}|a_0|\,|(s-z_{m'+1})|\cdots|(s-z_N)|$$

$$(2)\qquad |P_N(c)| \leqq M^{m'}|a_0|\,|(c-z_{m'+1})|\cdots|(c-z_N)|.$$

Since s is in K and $z_{m'+1}$, $\cdots$, z_N are all in J, we have

$$(3)\quad |(s-z_{m'+1})|\cdots|(s-z_N)| \geqq |(c-z_{m'+1})|\cdots|(c-z_N)|$$

* Such a region can be constructed by using the boundaries of the set of circles through c with centers within J but not within C'.

and upon combining (1), (2), and (3), we get

$$|P_N(s)| \geqq \frac{m^{m'}}{M^{m'}}|P_N(c)| > m^{m'}k/M^{m'},$$

since $|P_N(c)|>k$. This contradicts the fact that $|P_N(s)|< m^{m'}k/(4M^{m'})$ and thus yields our theorem.

COROLLARY I. *Every point of the circle* $|z|=1$ *is a limit point of the zeros of* $\{f(z)\}$.

Porter* gave an example which exhibited a set of convergents that converged over a region that extended outside of the circle of convergence of the power series. This example shows that the property that we have established for our convergents does not necessarily belong to every infinite subset of $\{f_n(z)\}$.

In conclusion, we point out that in the neighborhood of the circle $|z|=1$ the moduli of some of the convergents of the set $\{P_{n_i}(z)\}$ will take on any arbitrarily assigned positive value. In this respect the circle of convergence behaves very much like an essential singularity. However, there is one point of difference in that there are no Picard numbers associated with the circle of convergence.

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* Loc. cit., p. 191. Ostrowski (Journal of London Society, vol. 1 (1926), part 4) has recently made an extensive study of this "over-convergence." Ostrowski gives Jentzsch credit for having called attention to this property while in reality this credit should go to Porter, whose work antedates that of Jentzsch by ten years.

THE BINOMIAL QUARTIC AS A NORMAL FORM*

BY RAYMOND GARVER

From the standpoint of the Tschirnhausen transformation, the quartic equation may be said to occupy a distinct position, between the quadratic and cubic on one hand, and the higher degree equations on the other. For the lower degree equations can be reduced easily to binomial form by simple linear or quadratic transformations. Such reduction is not, in general, possible; but, for equations of degree not less than five, the second, third, and fourth terms can be removed by a fourth degree transformation without solving any equations of degree greater than the third.† (For the quintic such a transformation leads to the Bring-Jerrard normal form $x^5+a_4x+a_5=0$.) Now these two problems, reduction to binomial form, and removal of three intermediate terms, become equivalent for, and only for, the quartic equation. Hence the question of the reducibility of the quartic to the form $x^4+a_4=0$ is one of some interest.

It is worth while mentioning that none of the known methods for removing three terms apply to the quartic. They require, as mentioned above, fourth degree transformations, and fail when the given equation is of only that degree.‡

An examination of the literature shows that the question of the reducibility of the general quartic to binomial form by means of a Tschirnhausen transformation has been considered

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* Presented to the Society, October 29, 1927.

† See Dickson, *Modern Algebraic Theories*, 1926, Chap. XII; or almost any higher algebra.

‡ Salmon, *Modern Higher Algebra*, 4th ed., 1885, p. 250, and Matthiessen, *Grundzüge der Antiken und Modernen Algebra*, 1878, p. 125, state *incorrectly* that three terms can be removed from a quintic (or higher degree equation) by a third degree transformation, without solving any equations of degree greater than three.

by two men, and that they have stated different results. Lagrange, in his well known *Réflexions sur la Résolution Algébrique des Équations*,* gives an à priori proof that the reduction is possible. If we consider

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{1}$$

and the transformation

$$y = x^3 + k_2x^2 + k_3x + k_4, \tag{2}$$

the conditions that the new equation in y lack all intermediate terms are, as well known,

$$\Sigma y = 0, \quad \Sigma y^2 = 0, \quad \Sigma y^3 = 0,$$

where the summations extend over the four roots of (1). The equations of condition are of degree 1, 2, and 3 respectively in the parameters k_2, k_3, k_4, and if we eliminate any two of them, say k_3 and k_4, we shall arrive at a sixth degree equation. What Lagrange does, by an elegant though somewhat lengthy process, is to show that this sixth degree equation (when the original equation is a quartic) factors into three quadratics whose coefficients are themselves roots of cubic equations.

Over a hundred years later Sylvester, apparently unaware of this part of Lagrange's work, stated the opposite conclusion. On page 548 of his paper *On the so-called Tschirnhausen transformation*†, he says "Five is the minimum degree of equation from which three terms can be removed without solving an equation above the third degree," and a similar statement had been made earlier by Hamilton.‡ However, the proof of Sylvester is incomplete, or rather he is considering a slightly different problem, that of deter-

* Published in 1770–71. See his Oeuvres, vol. III, Paris, 1869, especially pp. 284–295.

† Journal für Mathematik, vol. 100 (1887), pp. 465–86. Collected Papers, vol. 4, Cambridge, 1912, pp. 531–49.

‡ Sixth Report of the British Association for the Advancement of Science, 1837, pp. 295–348.

mining in general how many parameters must be present to insure a solution of a system of equations without introducing any elevation of degree. He does not consider the possibility, in special cases, of a factorization such as the one established by Lagrange for the quartic.

I wish to indicate a reduction of the quartic to binomial form entirely different in form from that of Lagrange, and which, making use of some rather lengthy calculations which have been already worked out, seems much simpler. It makes use of a special type of Tschirnhausen transformation introduced by Hermite* and used later by Cayley,† in which the coefficients of the transformed equation are invariants. The use of the special transformation is by no means essential, but is very convenient because the transformed equation has been calculated, and been put in convenient form. Briefly, if we take our quartic in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0\,, \tag{3}$$

and apply the transformation

$$y = (ax + b)B + (ax^2 + 4bx + 3c)C + (ax^3 + 4bx^2 + 6cx + 3d)D, \tag{4}$$

then the second coefficient of the transformed equation will be zero, and the other coefficients will be invariants of the two forms $(a, b, c, d, e)(X, Y)^4$, $(B, C, D)(Y, -X)^2$.‡ The transformed equation has been computed by Cayley and Salmon;§ we do not need its explicit form. However we do need a result stated by Salmon;‖ namely, that the invariant

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* Comptes Rendus, vol. 46 (1858), p. 961; Oeuvres, vol. 2, Paris, 1908, pp. 30–37.

† *On Tschirnhausen's transformation*, Collected Papers, vol. 4, Cambridge, 1891, pp. 375–94. See also Journal für Mathematik, vol. 58 (1861), pp. 263–69.

‡ Cayley, loc. cit.

§ Their notations are different; I have followed Cayley above, but have left (5) in Salmon's notation.

‖ Loc. cit., p. 254.

$ace+2bcd-ad^2-eb^2-c^3$, which he calls T, becomes, for the new equation,

$$(5) \qquad T\phi^3 + S^2\Delta\phi^2 + 9\Delta^2 ST\phi + \Delta^3(54T^2 - S^3),$$

where

$$(6) \qquad S = ae - 4bd + 3c^2, \quad \Delta = (2/3)(BD - C)^2,$$

$$\phi = aB^2 + 4bBC + c(2BD + 4C^2) + 4dCD + eD^2.$$

If we denote (5) by T_y, the condition $T_y=0$ is a cubic in ϕ/Δ with known coefficients, which can then be solved. We then have to satisfy an equation of form $\phi/\Delta=r$, which, by (6), can be done by suitable choice of B, C, D. We then have the general quartic reduced to one in which the invariant T is zero.

Now it is known* that any quartic whose invariant T (known in algebraic geometry as the catalecticant) vanishes, can be reduced, by a linear fractional transformation (which is equivalent to a Tschirnhausen transformation), to binomial form. I have carried out the calculation, without any reference to the geometrical side of the problem, and while seemingly rather difficult it can be put in convenient form. Finally, since the product of two Tschirnhausen transformations is itself a Tschirnhausen transformation, we have the desired reduction.

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* Winger, *Introduction to Projective Geometry*, 1923, p. 209.

THE PROBABILITY LAW FOR THE INTENSITY OF A TRIAL PERIOD, WITH DATA SUBJECT TO THE GAUSSIAN LAW*

BY E. L. DODD

1. *Introduction.* Making use of Lord Rayleigh's† probability law for the resultant of n vibrations of fixed and equal amplitude but of chance phase, Schuster‡ found a probability law for the ratio of the square-root of the intensity of a trial period to the constant term of the Fourier development, and suggested the use of this as a criterion for the possible fortuitous nature of results apparently supporting the existence of periods under investigation. However, as periodicities in a sequence may be invariant to a change of origin, it seems desirable to have a criterion based upon intensity alone, which is thus invariant. Such a criterion will be derived in this paper. Some criterion—as Schuster pointed out—is indispensable in periodogram analysis. It signifies little that one trial period is more probable than another if all the apparent periodicities could be easily the result of chance. The question, then, is this : What fluctuations in the intensity may be expected when the data are subject to chance,—or, more specifically, to the Gaussian law?

2. *The Intensity of a Period and its Probability Law.* Suppose that the probability $p(x)$ that X_r will take on a value $\leqq x$ is given by

$$(1) \quad p(x) = \frac{h}{\pi^{1/2}} \int_{-\infty}^{x} e^{-h^2(t-b)^2} dt, \quad (r = 1,2, \cdots, km\ ;\ km = n),$$

* Presented to the Society, May 7, 1927.

† *On the resultant of a large number of vibrations of the same pitch and of arbitrary phase,* Philosophical Magazine, (5), vol. 10 (1880), pp. 73–78.

‡ *On lunar and solar periodicities of earthquakes,* Proceedings of the Royal Society of London, vol. 61 (1897), pp. 455–465.

with "precision" h. In testing for a period extending over k variates, suppose the data written in m rows of k elements each:

$$\begin{array}{lcccc} & X_1 & X_2 & \cdots & X_k \\ (2) & X_{k+1} & X_{k+2} & \cdots & X_{2k} \\ & \cdot & \cdot & \cdots & \cdot \\ \hline (3)\ \text{Averages} & Y_1 & Y_2 & \cdots & Y_k, \end{array}$$

where

$$Y_j = \frac{1}{m} \sum_{r=0}^{m-1} X_{j+kr}.$$

Taking $\theta = 2\pi/k$, set

$$S = S(k) = \sum_{r=0}^{k-1} Y_{r+1} \sin r\theta, \quad C = C(k) = \sum_{r=0}^{k-1} Y_{r+1} \cos r\theta,$$

and

$$(4) \qquad J = S^2 + C^2, \quad I = \frac{4}{k^2} J.$$

This quantity I will be called the intensity because, if, instead of supposing X_r governed by (1), we let $X_r = c \sin(r\theta + \alpha)$, thus making it strictly periodic, then $I = c^2$, when $k > 2$.

We note that J is a quadratic form, with determinant of elements

$$(5) \qquad a_{rs} = \cos(r - s)\theta, \qquad (r, s = 1, 2, \cdots, k).$$

By an orthogonal transformation, J may be reduced to a sum of squares, whose coefficients are the roots λ of the equation $F(\lambda) = 0$, obtained by subtracting λ from the elements of the main diagonal of the preceding determinant and equating this new determinant to zero. When $k > 2$, there are $k - 2$ zero roots; since, for any r and θ,

$$(6) \qquad \cos(r - 2)\theta - 2 \cos\theta \cdot \cos(r - 1)\theta + \cos r\theta = 0.$$

Indeed,* to the kth column add the $(k-2)$th column, and subtract the $(k-1)$th column multiplied by $2 \cos\theta$. The kth

*For this suggestion, the author is indebted to Miss Edna McCormick.

column becomes divisible by λ. Then take $r=(k-1)$, $(k-2), \cdots, 3$. But

$$(7)\qquad \begin{cases} (k-1)\cos^2\theta + (k-2)\cos^2 2\theta + \cdots \\ \qquad + \cos^2(k-1)\theta = \dfrac{k(k-2)}{4}, \quad \theta = \dfrac{2\pi}{k}, \end{cases}$$

is valid for $k>2$, and with the aid of this, it can be proved that

$$(8)\qquad F(\lambda) = (-1)^k\left\{\lambda^k - k\lambda^{k-1} + \frac{k^2}{4}\lambda^{k-2}\right\} = 0,$$

which has $k/2$ as a double root.

In (2), let us suppose the origin changed, and that X_r is now the new value obtained by subtracting b from the original X_r. This leaves S, C, J, and I unchanged, and its effect upon (1) is expressed by setting $b=0$. Using the corresponding new values of Y_r in (3), then, the probability that $y_r<Y_r<y_r+dy_r$ is

$$(9)\qquad \frac{h_1}{\pi^{1/2}}e^{-h_1^2y_r^2}dy_r, \qquad h_1 = hm^{1/2},$$

since by (3), each Y_r is the average of m values of X. By an orthogonal transformation, which does not change ΣY_r^2, we may now by (8) reduce J to

$$(10)\qquad J = \frac{k}{2}U^2 + \frac{k}{2}V^2 = u^2 + v^2,$$

where U and V follow the normal law with the same precision h_1, and u and v with precision $h_1(2/k)^{1/2}$. Then the probability that $z<J<z+dz$ is given by*

$$(11)\qquad h_2^2e^{-zh_2^2}dz, \qquad h_2^2 = 2h_1^2/k = 2mh^2/k.$$

To obtain the law for I in (5), we need only replace h_2^2 by

$$(12)\qquad H^2 = h_2^2k^2/4 = kmh^2/2 = nh^2/2.$$

* See Czuber, *Theorie der Beobachtungsfehler*, p. 148. Or Helmert, *Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler*. Zeitschrift für Mathematik und Physik, vol. 21 (1876), pp. 192–218.

THEOREM. *If each of $n=km$ independent variates X_r is subject to the Gaussian law* (1) *with precision h, and if from the averages Y_r of the columns of X's in* (2), *I is formed in accordance with* (4); *then, when $k>2$, the probability $P(z)$ that $I \geqq z$, is given by*

$$P(z) = e^{-nh^2z/2}. \tag{13}$$

In the case of $k=2$,

$$P(z) = \frac{2}{\pi^{1/2}} \int_{z'}^{\infty} e^{-t^2} dt, \qquad z' = \frac{h(nz)^{1/2}}{2}. \tag{14}$$

For the case of $k=2$, the result is easily obtained by rotating axes through 45°, noting that $I=(Y_1-Y_2)^2$.

3. *Application and Discussion.* The averaging process (3) preserves intact a period extending over k items, but tends to smooth out accidental variations and other periods. But this partial smoothing of accidental variations is by no means obliteration. When the value of I in (4) is computed, we ask: is it highly improbable that so large a value would be found if the data contained only chance fluctuations? For example, suppose that the standard deviation σ for 1000 variates under examination is 5, and by the usual rule we take $h^2=1/(2\sigma^2)=0.02$, $nh^2/2=10$. For some particular k, say $k=8$, suppose that $I=1$. By (13) the probability that a chance normal distribution would give as large a value as 1 is $P(1)=e^{-10}=0.000,05$. Normality granted, the evidence would support the belief that a period covering approximately 8 items exists in the data. The exact determination of this period is beyond the scope of this paper.

In the proof, it has not been assumed that S and C are independent. Such an assumption would, indeed, have led more quickly to the result (13). A slight plausibility for this assumption arises from the fact that the "expected" or "mean" value of SC is zero, if independently the variates are subject to the same arbitrary probability law with finite second moment.

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CONCERNING CONNECTED AND REGULAR POINT SETS*

BY G. T. WHYBURN

In this paper an extension will be given of a theorem of the author's† which states that if A and B are any two points of a continuous curve M, and K denotes the set of all those points of M which separate‡ A and B in M, then $K+A+B$ is a closed set of points. It will be shown that if A and B are any two points of any connected and regular (connected im kleinen) point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points. This extended theorem is applied to show that a simple continuous arc may be defined as a connected and regular point set which is irreducibly connected§ between some two of its points.

THEOREM 1. *If A and B are any two points of a connected and regular point set M, and K denotes the set of all those points of M which separate A and B in M, then $K+A+B$ is a closed and bounded set of points.*

PROOF. I shall first show that $K+A+B$ is closed. Suppose, on the contrary, that there exists a point P which does not belong to $K+A+B$ but which is a limit point of K. Then there exists a sequence of points $S=Y_1, Y_2, Y_3, \cdots,$

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* Presented to the Society, San Francisco Section, June 18, 1927.

† G. T. Whyburn, *Some properties of continuous curves*, this Bulletin, vol. 33 (1927), pp. 305–308.

‡ The points A and B of the connected point set M are said to be separated in M by the point X of M provided that $M-X$ is the sum of two mutually separated point sets containing A and B respectively.

§ A connected point set M is said to be irreducibly connected between two of its points A and B provided that no proper connected subset of M contains both A and B. See N. J. Lennes, American Journal of Mathematics, vol. 33 (1911), p. 308. See also Knaster and Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 206.

belonging to K and having P as its sequential limit point. Each point Y_i of this sequence separates M into two mutually separated sets $M_a(Y_i)$ and $M_b(Y_i)$ containing A and B respectively. Either there exists a subsequence W of S such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W, or there does not exist any such sequence.

CASE I. Suppose there exists a subsequence W of S such that if Y is any point of W, then $M_b(Y)$ contains at least one point of W. Let X_1 denote one of the elements of W. Then $M_b(X_1)$ contains at least one point of W. Let X_2 denote one such point of W which belongs to $M_b(X_1)$. Likewise $M_b(X_2)$ contains at least one point X_3 of W, and $M_b(X_3)$ contains at least one point X_4 of W, and so on. This process may be continued indefinitely, giving an infinite subsequence $V=X_1, X_2, X_3, \cdots$, of S. Now since for each positive integer i, $M_a(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} belongs to $M_b(X_i)$], therefore $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$. It follows that the points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Now let

$$E = \sum_{i=1}^{\infty} M_a(X_i), \qquad F = M - E.$$

Since for each positive integer i, $M=M_a(X_i)+X_i+M_b(X_i)$, and $M_a(X_i)+X_i$ is a subset of $M_a(X_{i+1})$, it is clear that $U=\sum_{i=1}^{\infty} X_i$ is a subset of E and that F is identical with the set of points common to all of the sets $M_b(X_i)$. Now no point of F, save possibly P (in case P belongs to M), is a limit point of E. For suppose some point Q of F, distinct from P, is a limit point of E. Then since Q is neither a point nor a limit point of U, there exists a circle C having Q as center and which neither contains nor encloses any point of U. But since M is regular at Q, and Q is a limit point of E, there exists a point Z of E which can be joined to Q by a connected subset I of M lying wholly within C. There exists an integer

k such that Z belongs to $M_a(X_k)$. But Q belongs to every set $M_b(X_i)$ and hence belongs to $M_b(X_k)$. But since I is connected and does not contain X_k, it follows that $M_a(X_k)$ and $M_b(X_k)$ are not mutually separated, contrary to hypothesis. Therefore no point of F, save possibly P, is a limit point of E. Furthermore, no point whatever of E is a limit point of F. For if Q is any point of E, then for some positive integer j, $M_a(X_j)$ contains Q, and since $M_b(X_j)$ contains F, and $M_a(X_j)$ and $M_b(X_j)$ are mutually separated, it follows that Q is not a limit point of F.

Now $M=E+F$, and if P does not belong to M, then E and F are mutually separated sets, contrary to the fact that M is connected. And if P does belong to M, then $M-P=E+(F-P)$ is the sum of two mutually separated sets E and $F-P$ containing A and B respectively. Hence P separates A and B in M and therefore belongs to K, contrary to supposition. Thus, in this case, the supposition that $K+A+B$ is not closed leads to a contradiction.

CASE II. Suppose S contains no subsequence W such that if Y is any element of W, then $M_b(Y)$ contains at least one point of W. Then no matter what subsequence W of S we take, W contains some point Y such that $M_a(Y)$ contains all the points of W except Y. Hence the sequence S itself contains a point X_1 such that $M_a(X_1)$ contains all the points of S except X_1. Let S_1 be the subsequence of S obtained by omitting the point X_1 from S. Then S_1 contains a point X_2 such that $M_a(X_2)$ contains all the points of S_1 except X_2. Let S_2 be the sequence obtained by omitting from S_1 the point X_2. Then S_2 contains a point X_3 such that $M_a(X_3)$ contains all the elements of S_2 except X_3. This process may be continued indefinitely, giving an infinite sequence of points $V=X_1, X_2, X_3, \cdots$, which is a subsequence of S. For each positive integer i, every point of the sequence S_i belongs to the set $M_a(X_i)$. Then since, for each i, $M_b(X_i)+X_i$ is connected and does not contain X_{i+1} [for X_{i+1} is an element of S_i, and hence belongs to $M_a(X_i)$], therefore $M_b(X_i)+X_i$ is a subset of $M_b(X_{i+1})$. It follows that the

points of the sequence V are all distinct, and hence P is the sequential limit point of this sequence.

Let

$$E = \sum_{i=1}^{\infty} M_b(X_i), \qquad F = M - E.$$

Then by an argument similar to that given under Case I, with the sets $M_a(X_i)$ and $M_b(X_i)$ interchanged, it is shown that in this case the supposition that $K+A+B$ is not closed leads to a contradiction. Therefore the set $K+A+B$ is closed.

I shall now show that the set $K+A+B$ is bounded. If K exists at all, it is clear that there exists a point O not belonging to M. Let T be an inversion of the plane about the point O. As M is regular, then $T(M)$ also is regular. Now $T(K)$ is identically the set of points in $T(M)$ which separate $T(A)$ and $T(B)$ in $T(M)$. Therefore, by the proof given above, $T(K)+T(A)+T(B)$ is a closed set of points. Since this set of points does not contain O, there exists a circle $T(C)$ with O as center and neither containing nor enclosing any point of the set $T(K)+T(A)+T(B)$. Then since the exterior of the circle $T(C)$ is the image of the interior of the circle C, the set of points $K+A+B$ lies wholly within the circle C and therefore is bounded. This completes the proof of the theorem.

THEOREM 2. *If the connected and regular point set M is irreducibly connected between some two of its points A and B, then M is a simple continuous arc from A to B.*

PROOF. Let K denote the set of all those points of M which separate A and B in M. Then since M is irreducibly connected between A and B, it follows by a theorem due to Knaster and Kuratowski* that $K=M-(A+B)$. Hence $K+A+B=M$. But by Theorem 1, $K+A+B$ is closed and bounded. Therefore M is closed and bounded and irre-

* *Sur les ensembles connexes*, loc. cit., p. 219, Theorem 19.

ducibly connected between A and B, and hence satisfies all the conditions of Lennes'* definition of a simple continuous arc from A to B.

The chief interest in the above definition of a simple continuous arc lies in the fact that in its statement nothing is said concerning the closure or the boundedness of the set in question. This result for the case of the arc is related to results obtained by R. L. Wilder† for the case of a simple closed curve. Wilder found that definitions of a simple closed curve could be given in which the closed and bounded conditions were replaced, or modified, by adding the condition of regularity.

THE UNIVERSITY OF TEXAS

NOTE ON THE FOURIER DEVELOPMENT OF CONTINUOUS FUNCTIONS‡

BY A. H. COPELAND

If we have given a function, $f(t)$, which is continuous and periodic, this function is not necessarily developable in a Fourier series, but by a monotone change of variable it becomes so. We shall prove in fact the following theorem.

THEOREM. *If a function, $f(t)$, is continuous and periodic with period $b-a$, then there exists a monotone continuous function, $t=t(\theta)$, transforming the interval $(0, 2\pi)$ into the interval*

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* N. J. Lennes, loc. cit. Hallett has shown that the condition of boundedness in Lennes' definition is superfluous. See G. H. Hallett, Jr., *Concerning the definition of a simple continuous arc*, this Bulletin, vol. 25 (1919), pp. 325–326. The same result was published two years later by Knaster and Kuratowski in their paper *Sur les ensembles connexes*, loc. cit., p. 224, Theorem 27.

† See an abstract of a paper *On the definition of simple closed curve*, this Bulletin, vol. 32 (1926), p. 123. See also p. 591.

‡ Presented to the Society, April 16, 1927.

*(a, b), such that $f[t(\theta)] = F(\theta)$ is developable in a uniformly convergent Fourier series in the interval $(0, 2\pi)$.**

Such a transformation does not alter the general character of the function $f(t)$.

The proof of this theorem depends upon a theorem of Fejér† to the effect that if a power series converges to a function $w(z)$ within the unit circle $|z| = 1$, and if $w(z)$ maps conformally the interior of the unit circle onto the interior of a region bounded by a closed Jordan curve, then the power series converges uniformly on the boundary of that circle.

We shall also have need of the following lemma.

LEMMA. *Given a function $\phi(t)$ which is continuous for $a \leqq t \leqq b$, and such that $\phi(a) = \phi(b) = 0$, then there exists a continuous function, $\sigma(t)$, which is of limited variation and such that $\sigma(a) = \sigma(b) = 0$ and $\phi(t) - \sigma(t) \geqq 0$ when $a < t < b$.*

Let m be the minimum of $\phi(t)$ in the interval (a, b). If $m \geqq 0$ we can let $\sigma(t) = 0$. Let us, then, consider the case $m < 0$. Let α and β, respectively, be the lower and the upper bounds of the set of roots of the equation $\phi(t) = m$. Let $\sigma(t)$ be the minimum of $\phi(t)$ in the interval (a, t) when $a \leqq t \leqq \alpha$, let $\sigma(t) = m$ when $\alpha < t < \beta$, and let $\sigma(t)$ be the minimum of $\phi(t)$ in the interval (t, b) when $\beta \leqq t \leqq b$. Then $\sigma(t)$ is seen to be continuous and to satisfy the conditions $\sigma(a) = \sigma(b) = 0$ and $\phi(t) - \sigma(t) \geqq 0$ for $a \leqq t \leqq b$. Moreover, $\sigma(t)$ is monotone in the intervals (a, α) and $\beta, b)$; and hence it is of limited variation in the interval (a, b).

If, instead of the function $\sigma(t)$, we take the function $\rho(t) = \sigma(t) - \sin[\pi(t-a)/(b-a)]$, then $\phi(t) - \rho(t) > 0$ when $a < t < b$. Furthermore $\rho(t)$ is continuous and of limited variation, and $\rho(a) = \rho(b) = 0$.

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* Suggested by G. C. Evans, *The Logarithmic Potential*, Colloquium Series, American Mathematical Society (in press). Chap. VII.

† See L. Fejér, *Über die Konvergenz der Potenzreihe an der Konvergenzgrenze in Fällen der konformen Abbildung auf die schlichte Ebene*, Mathematische Abhandlungen Hermann Amandus Schwarz zu seinem fünfzigjährigen Doktorjubiläum gewidmet.

Let us consider the function $f(t)$. Without loss of generality we may assume that $f(0)=f(2p)=0$, where $2p$ is the period of this function. Let

$$\phi(t) = f(t) - \frac{f(p)}{p}v(t),$$

where

$$v(t) = t, \quad \text{if } 0 \leqq t \leqq p,$$
$$v(t) = 2p - t, \quad \text{if } p \leqq t \leqq 2p.$$

Then $\phi(0)=\phi(p)=\phi(2p)=0$. It follows from the preceding lemma that there exists a function $\rho(t)$ which is of limited variation, continuous, and such that $u(t)=\phi(t)-\rho(t)$ is zero when $t=0$, p, or $2p$, and $u(t)$ is otherwise positive in the interval $(0, 2p)$.

Next let $u(t)=u_1(t)+u_2(t)$, where

$$\begin{cases} u_1(t) = u(t), & \text{if } 0 \leqq t \leqq p, \\ u_1(t) = 0, & \text{if } p \leqq t \leqq 2p; \end{cases}$$

$$\begin{cases} u_2(t) = 0, & \text{if } 0 \leqq t \leqq p, \\ u_2(t) = u(t), & \text{if } p \leqq t \leqq 2p. \end{cases}$$

The pairs of functions $[u_1(t), v(t)]$ and $[u_2(t), v(t)]$ define two closed Jordan curves, C_1 and C_2.

The region interior to C_1 can be mapped conformally onto the interior of the unit circle, $|z|=1$. Between the points of C_1 and the points of the unit circle, there is a one-to-one and continuous correspondence preserving cyclic order.* That is, there exists a continuous monotone function, $t=t_1(\theta)$, relating the points, $z=e^{i\theta}$, of the unit circle to the points, $w=u_1(t)+iv(t)$, of the curve C_1. In particular, the mapping may be accomplished in such a manner that the points $\theta=0$, $\theta=\pi$ correspond respectively to the points $t=0$, $t=p$. Then $t_1(0)=0$ and $t_1(\pi)=p$.

* See C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis*, Mathematische Annalen, vol. 73 (1913), pp. 305–320.

Let $w=w_1(z)$ be the mapping function. Then $w_1(z)$ can be expanded in a power series about the origin. This series converges when $|z|<1$. Furthermore, by the theorem of Fejér, this series converges uniformly on the boundary of the circle $|z|=1$. It follows that the function $U_1(\theta)=u_1[t_1(\theta)]$ is developable in a uniformly convergent Fourier series.

In a like manner it can be seen that there exists a function $t=t_2(\theta)$ which is monotone, continuous, and such that $t_2(0)=0$ and $t_2(\pi)=p$, and also such that the function $U_2(\theta)=u_2[t_2(\theta)]$ is developable in a uniformly convergent Fourier series.

The sum $U_1(\theta)+U_2(\theta)$ defines a function $U(\theta)$ which is developable in a uniformly convergent Fourier series in the interval $(0, 2\pi)$, and we shall show that there exists a monotone continuous function $\theta=\theta(t)$ such that $U[\theta(t)]\equiv u(t)$. To do this, let us define a function $t(\theta)$ by the equations

$$\begin{cases} t(\theta) = t_1(\theta), & \text{if } 0 \leqq \theta \leqq \pi, \\ t(\theta) = t_2(\theta), & \text{if } \pi \leqq \theta \leqq 2\pi, \end{cases}$$

and let the inverse of $t=t(\theta)$ be $\theta=\theta(t)$. Then these two functions are continuous and monotone. Moreover

$$U_1[\theta(t)] + U_2[\theta(t)] = u_1(t), \quad \text{when } 0 \leqq t \leqq p,$$
$$U_1[\theta(t)] + U_2[\theta(t)] = u_2(t), \quad \text{when } p \leqq t \leqq 2p,$$

since $U_1(\theta)=0$ in the interval $(\pi, 2\pi)$, and $U_2(\theta)=0$ in the interval $(0, \pi)$. Thus $U_1[\theta(t)]+U_2[\theta(t)]\equiv u(t)$. Furthermore the functions $v[t(\theta)]$ and $\rho[t(\theta)]$ are of limited variation with respect to θ. Therefore

$$F(\theta) = U_1(\theta) + U_2(\theta) + \rho[t(\theta)] + \frac{f(p)}{p} v[t(\theta)]$$

is developable in a uniformly convergent Fourier series. But $F(\theta)=f[t(\theta)]$; hence the theorem is established.

THE RICE INSTITUTE

ZEROS OF A FUNCTION AND OF ITS DERIVATIVE*

BY W. J. TRJITZINSKY

Macdonald has proved the following theorem.† Let $f(z)=u(x, y)+iv(x, y)$ be a function of z analytic throughout the interior of a single closed curve C, defined by the equation $|f(z)|=M$, where M is a constant. Let C be an ordinary curve in the sense that if ψ be the angle which the tangent to C makes with the x-axis, and if the point of contact of the tangent describes the curve C, ψ will increase by 2π. Then the number of zeros of $f(z)$ in this region exceeds the number of zeros of the derivative $f'(z)$ by unity. Our purpose is to generalize this result by showing the conclusion holds true even when M is not a constant, but a function $M=M(x, y)$ analytic and greater than zero within sufficiently large intervals, and C is a curve single, closed, and ordinary (in the sense mentioned above), and satisfying the equation $|f(z)|=M(x, y)$, provided that the point $(x=x(\theta), y=y(\theta))$ describes C once when θ changes from 0 to 2π; where $x(\theta)$ and $y(\theta)$ are periodic functions satisfying the equations

$$(1) \qquad \frac{u(x,y)}{M(x,y)}=\cos\theta, \qquad \frac{v(x,y)}{M(x,y)}=\sin\theta.$$

The solvability of these equations implies that the Jacobian of the left members does not vanish identically.

On C, let $f(z)=M(x, y)e^{i\theta}$. Then there results a pair of equations (1) whose solutions are $x_i=x_i(\theta)$ and $y_i=y_i(\theta)$ of period 2π in θ. By hypothesis, among them there is at least one solution, say $x=x(\theta)$, $y=y(\theta)$, representing the curve C, and the point $(x=x(\theta), y=y(\theta))$ describes C once as θ varies from 0 to 2π.

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* Presented to the Society, San Francisco Section, June 18, 1927.

† Proceedings of the London Society, vol. 29 (1898), pp. 576–577; Proceedings of the London Society, (2), vol. 15 (1916), pp. 227–242.

Letting $M(x(\theta), y(\theta)) = N(\theta)$, we observe that $N(\theta)$ as well as $N'(\theta)$ is of period 2π, and for all θ, $N(\theta) > 0$, so that when z describes C and θ varies from 0 to 2π, the complex quantity $N'(\theta) + iN(\theta)$ describes a closed curve entirely above the real axis; its modulus will return to the initial value and the variation of its argument will be zero. Now on C we have

$$f(z) = N(\theta) \cdot e^{i\theta},$$

$$f'(z) = e^{i\theta} \cdot \frac{d\theta}{dz} \cdot [N'(\theta) + iN(\theta)],$$

$$f''(z) = e^{i\theta} \cdot f\Big\{ [N'(\theta) + iN(\theta)] \cdot \left[\frac{d^2\theta}{dz^2} + i\left(\frac{d\theta}{dz}\right)^2\right] + [N''(\theta) + iN'(\theta)]\left(\frac{d\theta}{dz}\right)^2 \Big\}.$$

The excess e of the number of zeros of $f(z)$ over the number of zeros of $f'(z)$ within C is

$$e = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\int_C \frac{f''(z)}{f'(z)}dz$$

$$= -\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)}dz - \frac{1}{2\pi i}\int_C \frac{d\theta}{dz}\left[\frac{N''(\theta) + N'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz.$$

The quantity

$$-\frac{1}{2\pi i}\int_C \frac{\left(\frac{d^2\theta}{dz^2}\right)}{\left(\frac{d\theta}{dz}\right)} \cdot dz = 1,$$

as can be seen from Whittaker's* presentation of the theorem cited. Hence we have

* Whittaker and Watson, *Modern Analysis*, 3d ed., p. 121.

$$e = 1 - \frac{1}{2\pi i}\int_C \left(\frac{d\theta}{dz}\right)\cdot\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz$$

$$= 1 - \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]d\theta,$$

since θ varies from 0 to 2π when z describes C. Moreover

$$e = 1 - \frac{1}{2\pi i}\cdot\{\log[N'(\theta) + iN(\theta)]\}_0^{2\pi} + \frac{1}{2\pi i}[\log N(\theta)]_0^{2\pi}.$$

We know that $N(\theta)$ is real; hence $[\log N(\theta)]_0^{2\pi} = 0$. On the other hand, the variation of the argument of $[N'(\theta) + iN(\theta)]$ as θ changes from 0 to 2π is zero, so that $\log[(N'(\theta) + iN(\theta))]_0^{2\pi} = 0$. Hence $e = 1$. This proves the theorem.

THE UNIVERSITY OF TEXAS

LINEAR INEQUALITIES IN GENERAL ANALYSIS*

BY L. L. DINES

1. *Introduction.* In his studies in general analysis, E. H. Moore has developed† a theory of the linear functional equation

$$\xi + J\kappa\xi = \eta.$$

Here ξ and η denote functions (the latter given, the former to be determined) belonging to a class $\mathfrak{M}$ of real-valued functions on a general range $\mathfrak{P}$. The kernel function κ belongs to a class $\mathfrak{K}$ which is well defined in terms of the fundamental class $\mathfrak{M}$. A sufficient foundation for the theory is laid by means of postulates upon the class $\mathfrak{M}$ and the functional operation J.

The purpose of the present paper is to consider the linear *inequality*

$$\xi + J\kappa\xi > 0, \tag{1}$$

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* Presented to the Society, September 8, 1927.

† *On the foundations of the theory of linear integral equations*, this Bulletin, vol. 18, pp. 334–362.

a type which includes the *non*-homogeneous inequality in a sense indicated in § 5.

The most interesting feature is the relationship between the inequality (1) and the *adjoint equation*

$$\mu + J\mu\kappa = 0,$$

as expressed in the theorem of § 4. For the proof of this theorem, two postulates are introduced in addition to those used by Professor Moore.

2. *The Basis.* As a basis we assume Moore's $\sum_5$:*

$$(\mathfrak{A}\ ;\ \mathfrak{P}\ ;\ \mathfrak{M}\ ;\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*\ ;\ J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

where $\mathfrak{A}$ is the class of real numbers; $\mathfrak{P}$ is a general class of elements p; $\mathfrak{M}$ is a class of functions (μ, ξ, η, π) on $\mathfrak{P}$ to $\mathfrak{A}$, having the properties L, C, D; and the operator J has the properties L and M.

3. *An Equation Equivalent to the Inequality.* The inequality (1) is obviously equivalent to an equation

$$\xi + J\kappa\xi = \pi, \tag{2}$$

where π is any function of $\mathfrak{M}$ which is everywhere positive on $\mathfrak{P}$. This situation suggests an additional postulate upon the class $\mathfrak{M}$, namely that it contain at least one positive function. We shall find it desirable presently to make a more comprehensive postulate.

Equation (2) is of the Fredholm type treated by Moore. Its solution depends upon the Fredholm determinant F_κ of the kernel κ. In case $F_\kappa \neq 0$, the equation admits the solution

$$\xi = \pi + J\lambda\pi, \tag{3}$$

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* Loc. cit., p. 349. We might have assumed the basis $\sum_6$ (p. 352) which differs from $\sum_5$ in the presence of two classes of functions $\mathfrak{M}'$ and $\mathfrak{M}''$ on two conceptually distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$. The basis $\sum_6$ reduces to $\sum_5$ if $\mathfrak{P}' = \mathfrak{P}''$ and $\mathfrak{M}' = \mathfrak{M}''$. Our choice is in the interest of typographical simplicity. We were influenced also by the fact that we have discovered no instance involving distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$ in which our postulate E (§4) is satisfied.

where λ is the reciprocal kernel (resolvent) for κ. The inequality (1) then admits, in this case, the solution (3) where π is any positive function of $\mathfrak{M}$.

Of more interest is the exceptional case in which $F_\kappa = 0$. The theory of equation (2) for this case (without the restriction that π be positive) has been treated in detail by Hildebrandt.* The facts are entirely analogous to those well known in the theory of ordinary integral equations. The equation (2) in general admits no solution. The corresponding homogeneous equation

$$\xi + J\kappa\xi = 0$$

admits a finite number of linearly independent solutions $\xi_1, \xi_2, \cdots, \xi_m$; and the adjoint homogeneous equation

$$\mu + J\mu\kappa = 0 \tag{4}$$

admits the same number of linearly independent solutions $\mu_1, \mu_2, \cdots, \mu_m$. A necessary and sufficient condition that the equation (2) admit a solution is that π be orthogonal to each of the functions μ_i. That is, that

$$J\mu_i\pi = 0, \qquad (i = 1,2, \cdots, m). \tag{5}$$

The solution of (2) is then

$$\xi = \pi + J\lambda\pi + \sum_{i=1}^{m} c_i\xi_i$$

where λ is a pseudo-resolvent for κ, and the c_i are arbitrary constants.

To apply these results to the inequality (1), we note that the kernel κ is given; and if $F_\kappa = 0$, the set of fundamental solutions $\mu_1, \mu_2, \cdots, \mu_m$ of the adjoint equation (4) can be determined. The problem of the inequality then reduces to the determination of a positive function π in $\mathfrak{M}$ which satisfies conditions (5).

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* *On pseudo-resolvents of linear integral equations in general analysis*, Annals of Mathematics, vol. 21 (1920), pp. 323–330.

The machinery which suffices for the theory of the functional equation apparently affords no means of attack on this problem. We therefore introduce in the next section new postulates which are sufficient to secure our desired result.

4. *Two New Postulates.* The postulates to be introduced involve a certain property relative to functions, which has been used in an earlier paper,* and which may be defined as follows:

A real-valued function is said to be *M-definite* if it is somewhere positive and nowhere negative, or somewhere negative and nowhere positive on its range; in other words if it is not identically zero and does not change sign.

Postulate N: If μ is an M-definite function of $\mathfrak{M}$ and π is a positive function of $\mathfrak{M}$, then $J\mu\pi \neq 0$.

Postulate E: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of $\mathfrak{M}$ such that no linear combination of them is M-definite, then there exists a positive function π in $\mathfrak{M}$ such that

$$J\mu_i\pi = 0, \qquad (i = 1,2, \cdots, m). \tag{5}$$

The postulate N is a natural generalization of a simple property possessed by the operator J in each of Moore's classical instances (I), (II), (III), and (IV).

The existential postulate E is not perhaps so natural, and it undoubtedly invites further analysis. It is however satisfied in each of the four classical instances, as has been shown elsewhere by the author.† This postulate (together

* See *On sets of functions of a general variable*, Transactions of this Society, vol. 29 (1927), pp. 463–470.

† For the instance (II), including (I), see *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, vol. 28 (1926), p. 41, Theorem II.

For the instance (III), the proof is contained in a paper entitled *A theorem on orthogonal sequences* which will probably be published at an early date.

For the instance (IV), see *A theorem on orthogonal functions with an application to integral inequalities*, which is to appear soon in the Transactions of this Society.

An equivalent (and in some respects preferable) formulation of the postulate may be given in terms of the notion M-rank defined in the paper

with the assumption that $\mathfrak{M}$ has the property L) implies the existence of at least one positive function in $\mathfrak{M}$.

5. *The Inequality and its Adjoint Equation.* By use of the postulates just introduced, we now obtain the following

THEOREM: *The inequality*

$$\xi + J_K\xi > 0 \tag{1}$$

admits a solution, if and only if the adjoint equation

$$\mu + J\mu K = 0 \tag{6}$$

admits no M-definite solution.

First, we note that the case $F_K \neq 0$ is consistent with the theorem, since in that case the inequality (1) admits a solution (3), and the equation (6) admits only the solution $\mu = 0$ which is not M-definite.

Suppose then that $F_K = 0$, and consider a fundamental set of solutions

$$\mu_1, \mu_2, \cdots, \mu_m \tag{7}$$

of the adjoint equation (6). That is, the linear combinations of this set comprise the totality of solutions of (6). There are then two possibilities.

If the set (7) admits no linear combination which is M-definite, then the equation (6) admits no M-definite solution, while the inequality (1) admits a solution, since by postulate E there is a positive function π satisfying (5).

If, on the other hand, the set (7) admits an M-definite linear combination, say $\mu^* = \sum c_i\mu_i$, then the equation (6) admits the M-definite solution $\mu = \mu^*$. And the inequality (1) admits no solution. For suppose it did admit a solution $\xi = \xi^*$, that is suppose $\xi^* + J_K\xi^* = \pi$, where π is a positive function. Then π must satisfy the conditions $J\mu_i\pi = 0$,

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referred to in the preceding footnote, namely: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of M-rank zero, then there exists a positive function π in $\mathfrak{M}$ such that $J\mu_i\pi = 0$, $(i = 1, 2, \cdots, m)$.

and since J has the property L (linearity) it follows that $J\mu^*\pi = 0$, which contradicts postulate N.

We have shown that (1) admits a solution in those and only those cases in which (6) admits no M-definite solution. This establishes the theorem.

6. *The Non-Homogeneous Inequality.* The inequality

$$\xi + J\kappa\xi > \eta \tag{8}$$

where η is a given function of $\mathfrak{M}$, may be replaced by the two simultaneous inequalities

$$\begin{aligned} \xi + J\kappa\xi - \eta x &> 0, \\ x &> 0, \end{aligned} \tag{9}$$

where x is a real number to be determined. For any solution $\xi = \xi^*$ of (8) determines a solution ($\xi = \xi^*$, $x = 1$) of (9), and conversely any solution ($\xi = \xi^*$, $x = x^*$) of (9) determines a solution $\xi = \xi^*/x^*$ of (8).

By the method of adjunctional composition (see Moore, loc. cit., p. 355), the system (9) may be written in the homogeneous form

$$\xi' + J\kappa'\xi' > 0$$

relative to a new basis $\sum_5'$:

$$(\mathfrak{A} \,;\, \mathfrak{P}' \,;\, \mathfrak{M}' \,;\, \mathfrak{K}' \equiv (\mathfrak{M}'\mathfrak{M}')^* \,;\, J' \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}).$$

This new basis is the adjunctional composite of our general basis $\sum_5$, and the particular and very simple basis

$$(\mathfrak{A} \,;\, \mathfrak{P}^{\mathrm{I}} \,;\, \mathfrak{M}^{\mathrm{I}} \equiv \mathfrak{A} \,;\, \mathfrak{K}^{\mathrm{I}} \equiv \mathfrak{A}\mathfrak{A} \,;\, J^{\mathrm{I}} \text{ on } \mathfrak{K}^{\mathrm{I}} \text{ to } \mathfrak{A}),$$

in which $\mathfrak{P}^{\mathrm{I}}$ is a class containing a single element, $\mathfrak{M}^{\mathrm{I}}$ is the class of real numbers, and J^{I} is the identity operation.

UNIVERSITY OF SASKATCHEWAN

SUBSTITUTIONS WHICH TRANSFORM A REGULAR GROUP INTO ITS CONJOINT*

BY G. A. MILLER

In a recent article which appeared in this Bulletin† under a similar heading it was proved that a necessary and sufficient condition that there exists a substitution t of order 2 which is commutative with every substitution of the group of isomorphisms of a regular group R, transforms R into its conjoint, and involves only letters of R, is that R involves substitutions whose order exceeds 2. In what follows it will be assumed that R satisfies this condition, and hence R may represent any abstract group except the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. While in the previous article the main emphasis was laid on a method of determining t as a substitution on the letters of R, and comparatively little attention was then paid to the fact that t is commutative with every substitution of the group of isomorphisms of R, it is this fact which will receive the main attention in the present article, since it throws much light on the infinite system of substitution groups which are separately groups of isomorphisms of regular substitution groups.

To simplify the considerations which follow we note here the well known facts that the totality of the substitutions on the letters of R which transform R into itself, in the sense that they transform every substitution of R into such a substitution, constitutes a group H, known as the holomorphs of R, and that the subgroup H_1, composed of all the substitutions of H which omit a given letter, is simply isomorphic with the group of isomorphisms of R. There are always at least two such sub-groups in H, and t is not necessarily commutative with every substitution in each of

* Presented to the Society, September 9, 1927.

† Vol. 32 (1926), p. 631.

these subgroups. It must, however, be commutative with every substitution of such a subgroup whenever t transforms into their inverses all the cycles which involve a letter which is neither in t nor in the subgroup in question. In what follows such a subgroup will be implied when we speak of the group of isomorphisms of R in connection with t.

When R is abelian then t is one of the substitutions of H which transform every substitution of R into its inverse and omit at least one letter of R. In this case the degree of t cannot be less than $r/2$, where r is the order of R. On the other hand, the degree of t is $r/4$ whenever R is the direct product of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. Since this infinite category of groups is characterized by the fact that it is composed of all the groups which involve operators whose order exceeds 2 but the number of such operators is relatively less than in any other group,* it is clear that t is never of a smaller degree than $r/4$, and that it is of this degree only when R belongs to the category just noted. From the obvious fact that when R is the quaternion group it can be transformed into its conjoint by a transposition, while t is of degree 6 in this case, it results directly that it may sometimes be possible to transform R into its conjoint by means of a substitution which is of a lower degree than t. In what follows we shall use t_1 to represent a substitution of lowest degree which transforms R into its conjoint, except that t_1 cannot be the identity when R is abelian and therefore coincides with its conjoint.

It is easy to see that the degree of t_1 can never be less than $r/4$ since the product of t_1 and any of its conjugates under H must be found in H, in view of the fact that every substitution which transforms R into its conjoint must also transform this conjoint into R. Moreover, the degree of this product cannot be less than $r/2$, whenever this product is not the identity, since the number of letters omitted by any substitution of H_1 is equal to the number of substitutions

* G. A. Miller, Comptes Rendus, vol. 141 (1905), p. 591.

of R with which this substitution is commutative and hence H cannot involve any substitution besides the identity which involves less than $r/2$ letters. From this it results also that when t_1 is of degree $r/4$ then R must be non-abelian, and that a conjugate of t_1 cannot involve the same letters as t_1 involves unless it is identical with t_1. Hence, whenever t_1 is of degree $r/4$, it is of order 2 and has exactly four distinct conjugates under H. The continued product of these conjugates is invariant under H and hence is a characteristic substitution of R. In what follows we shall assume that t_1 is of degree $r/4$.

If H_1 omits a letter of t_1 all of its substitutions must be commutative with t_1 and hence t_1 is also commutative with every substitution of the group of isomorphisms of R. It therefore results that when R can be transformed into its conjoint by a substitution of degree $r/4$ and t is not of this degree then there are at least three different substitutions of order 2 which are commutative with every substitution in its group of isomorphisms. Since t_1 is commutative with one-fourth of the substitutions of R and transforms R into its conjoint it results that the central of R must be composed of one-fourth of its substitutions. Since the central quotient group is always non-cyclic it results that R involves three abelian subgroups of index 2. Hence the following theorem

THEOREM I. *A regular group cannot be transformed into its conjoint by a substitution which is not the identity but involves less than one-fourth of the letters of this regular group, and when it is transformed into its conjoint by a substitution which involves exactly one-fourth of its letters its order must be divisible by* 8 *and it must be non-abelian but involve three abelian subgroups of index* 2.

We proceed to prove that the conditions expressed in this theorem are not only necessary but also sufficient in order that the regular group R can be transformed into its conjoint by t_1. When these conditions are satisfied R has a commutator subgroup of order 2. Let R_0 be one of its abelian subgroups

of index 2. This subgroup has two transitive constituents and involves the central of R, which has four such constituents. Let t_1 be the part of the commutator of R which appears in the last of these four constituents and let s be the substitution of order 2 which simply interchanges the corresponding letters of R_0, so that s is commutative with every substitution of R_0 and transforms t_1 into t_0. To obtain R it is obviously only necessary to extend R_0 by means of the substitution $s_0 t_0 t_1 s$, where s_0 is a substitution found both in the first transitive constituent of R_0 and also in the central of R.* The transform of R under t_1 is the conjoint of R since each of its substitutions is commutative with every substitution of R. In fact t_1 is commutative with one-half of the substitutions of R_0 and transforms each of the other substitutions of R_0 into itself multiplied by the substitution of order 2 in the second transitive constituent of R_0 which includes t_1. These transforms are obviously commutative with $s_0 t_0 t_1 s$. As the transform of $s_0 t_0 t_1 s$ under t_1 is also commutative with every substitution of R we have proved the following theorem.

THEOREM II. *A necessary and sufficient condition that a regular group can be transformed into its conjoint by a substitution which involves exactly one-fourth of its letters is that this group be non-abelian but involve more than one abelian subgroup of index* 2.

From this theorem it results that there are at least three substitutions of order 2 on the letters of the H of each of these groups which are commutative with every substitution of H_1, except possibly in the case when R is the direct product of the octic group and an abelian group of order 2^m and of type $(1, 1, 1, \cdots)$. In this special case it is obvious that there are always exactly three such substitutions in H. Hence it results that when the regular group R is non-abelian but involves more than one abelian subgroup of index 2 there

* G. A. Miller, American Journal of Mathematics, vol. 24 (1902), p. 395.

are always at least two distinct substitutions of order 2 which transform it into its conjoint and are commutative with every substitution of its group of isomorphisms. It also results directly from the same theorem that the only non-abelian groups which can be transformed into their conjoints by a transposition are the octic group and the quaternion group. It may be added that when R is any dihedral group of order $2n$, $n>2$, then t is a substitution of order 2 which transforms into its inverse a cycle of order n and involves $n-1$ or $n-2$ letters as n is odd or even. When R is the dicyclic group then t is the product of this substitution and the substitution of order 2 generated by the second cycle of order n.

If all the operators of order $k>2$ which appear in a group G are transformed according to a transitive substitution group under the group of isomorphisms of G then there are at least $\phi(k)$ substitutions on the letters of this transitive group which are commutative with each of its substitutions, $\phi(k)$ being the totient of k. This results from the fact that the subgroup composed of all the substitutions of this transitive group which omit one letter must omit at least the $\phi(k)$ letters corresponding to the operators of order k generated by a group operator of this order. In particular, there results the following theorem.

THEOREM III. *A necessary and sufficient condition that a constituent of t which involves the same letters as a transitive constituent of H involves appears in this transitive constituent is that each of the operators of G which corresponds to the letters of this constituent corresponds also to its inverse in some automorphism of G.*

This theorem explains the fact that the constituents of t appear in H when R is dihedral or dicyclic.

According to the general theorem under consideration there is always at least one substitution of order 2 on the letters of H which is commutative with every substitution

of H_1, and when there is only one such substitution it must be t. It may be of some interest to inquire what condition R must satisfy in order that there be only one such substitution. When R is abelian and $r>4$ the necessary and sufficient condition is obviously that the order of R be a power of any odd prime number and that its type be $(1, 1, 1, \cdots)$. If such an abelian group is extended by an operator of order 2 which transforms each of its operators into its inverse, there results a generalized dihedral group whose group of isomorphisms is the holomorph of this abelian group and hence t is again the only substitution of order 2 on the letters of its H which is commutative with every substitution of H_1. Another infinite system of groups in which t is the only substitution on the letters of R which is commutative with every substitution of H_1 may be constructed by extending the abelian group of order 2^m and of type $(1, 1, 1, \cdots)$ by means of an operator of prime order which is generated by any operator of order 2^m-1 in the group of isomorphisms of this abelian group. The tetrahedral group is the group of lowest order in this system. From what was noted above it results directly that whenever the order of a regular group is divisible by at least two distinct odd prime numbers there must be more than one substitution on the letters of R which is commutative with every substitution of its H_1.

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THE FUNCTIONAL EQUATION DEFINING DIOPHANTINE AUTOMORPHISMS*

BY LOUIS WEISNER†

1. *Introduction.* A form $f(x_1, \cdots, x_m)$ is said to admit a *diophantine automorphism* if there exist m forms

$$\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m),$$

such that

$$f[\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m)] = f^n(x_1, \cdots, x_m), \tag{1}$$

where n is a positive integer greater than 1, the coefficients of the forms being integers or rational integral functions of one or more parameters with integral coefficients. In view of Bell's remarks (*A diophantine automorphism*, this Bulletin, vol. 28, p. 71) on the scarcity of what he calls *proper* diophantine automorphisms, it seems desirable to seek solutions of (1), regarded as a functional equation, no restrictions being placed on the algebraic nature of the solution or the coefficients.

Three methods of obtaining solutions of this functional equation are offered in the present paper. The first method gives a solution corresponding to any one-parameter continuous group having certain properties. The solution is generally a transcendental function, but I present this method here as the functional equation is of interest in analysis. In the second method f is assumed known, and the ϕ's are to be determined. It is shown that if there exists a Cremona transformation of which $x_i' = f(x_1, \cdots, x_m)$ is a constituent, then there exist *rational* functions $\phi_1, \cdots, \phi_m$ satisfying (1). This leads to rational solutions of the diophantine equation

$$f(x_1, \cdots, x_m) = u^n.$$

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† National Research Fellow.

The third method is more satisfactory for the purposes of diophantine analysis. It is shown that if the ϕ's are forms such that the equations

$$x_i' = \phi_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m),$$

define a Cremona transformation of finite period, then there exists a form $f(x_1, \cdots, x_m)$ satisfying (1). Eisenstein's automorphism is obtained in this way. A diophantine automorphism may be obtained from every Cremona transformation of finite period, but not every diophantine automorphism may be obtained in this way. For example, the transformation

$$x_1' = x_1^n, \cdots, x_m' = x_m^n$$

is not a Cremona transformation, but it transforms $x_1 x_2 \cdots x_m$ into its nth power.

2. *First Method. One-Parameter Groups.* Let

$$x_i' = \phi_{it}(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m) \tag{2}$$

be a one-parameter continuous group (t being the parameter) analytic in the neighborhood of $t=0$ and in some region of the x-space. The variables are non-homogeneous. We assume that the parameter of the product of two transformations of the group equals the sum of the parameters of the transformations, and that the identical transformation is given by $t=0$.*

Instead of (1) we consider

$$f(\phi_{1t}, \cdots, \phi_{mt}) = f^{n^t}(x_1, \cdots, x_m), \tag{3}$$

which reduces to (1) when $t=1$. Taking the logarithms of both members of (3) we obtain

$$g(\phi_{1t}, \cdots, \phi_{mt}) = n^t g(x_1, \cdots, x_m), \tag{4}$$

where

$$g(x_1, \cdots, x_m) = \log f(x_1, \cdots, x_m). \tag{5}$$

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* See A. Cohen, *The Lie Theory of One-Parameter Groups*, §4.

Differentiating (4) with respect to t, and then putting $t=0$, we obtain

$$Ug(x_1, \cdots, x_m) = g(x_1, \cdots, x_m) \log n, \tag{6}$$

where U denotes the linear differential operator defining the infinitesimal transformation of the group (2). From (6) we deduce that

$$U^p g = g^p \log n.$$

Hence

$$\begin{aligned} g[\phi_{1t}, \cdots, \phi_{mt}] &= g + tUg + \frac{t^2}{2!}U^2 g + \cdots \\ &= g + tg \log n + \frac{t^2}{2!}\log^2 n + \cdots \\ &= gn^t. \end{aligned}$$

It follows that every solution of (6) is a solution of (4) which leads to a solution of (3) through the medium of (5). Now (6) is a linear partial differential equa ion of Lagrange's type which may be solved by the usual methods, the solution involving an arbitrary function. A solution of (1) may thus be obtained from any continuous group satisfying the stated conditions.

3. *Second Method. Arbitrary Transformations.* If

$$x_i' = f_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m), \tag{7}$$

the variables being again non-homogeneous, is a reversible, but otherwise arbitrary transformation, the equations

$$f_i(x_1', \cdots, x_m') = F_i(x_1, \cdots, x_m), \quad (i = 1, \cdots, m), \tag{8}$$

where F_i is arbitrary, can be solved for $x_1', \cdots, x_m'$, giving, say,

$$x_i' = G_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m). \tag{9}$$

Evidently

$$f_i(G_1, \cdots, G_m) = F_i(x_1, \cdots, x_m), \quad (i = 1, \cdots, m). \tag{10}$$

Regarding (10) as a system of functional equations in which $G_1, \cdots, G_m$ are the unknowns, a solution is provided by (9). If $h<m$ equations of the type (10) are given, we add $m-h$ other equations and obtain a solution involving $m-h$ arbitrary functions. Thus to find solutions of (1), f being given, select $m-1$ arbitrary functions $f_2, \cdots, f_m$, such that (7) is reversible ($f_1 \equiv f$), and let $F_2, \cdots, F_m$ be arbitrary functions. We obtain a solution involving $F_2, \cdots, F_m$.

Where f is a given *rational* function it is not always possible to find functions $f_2, \cdots, f_m$ such that $\phi_1, \cdots, \phi_m$ are rational. This will be the case, however, if (7) is a Cremona transformation, one of whose right members is f.

4. *An Example.* Applying the method of §3 to the equations

$$x' = \frac{x}{x^2 + y^2}, \qquad y' = \frac{y}{x^2 + y^2},$$

defining an inversion, we obtain the equations

$$\frac{x'}{x'^2+y'^2} = \left(\frac{x}{x^2 + y^2}\right)^n, \qquad \frac{y'}{x'^2+y'^2} = F,$$

where F is an arbitrary rational function of x and y. Solving for x' and y', we obtain

$$x' = \frac{x^n(x^2 + y^2)^n}{x^{2n} + (x^2 + y^2)^{2n}F^2},$$

$$y' = \frac{(x^2 + y^2)^nF}{x^{2n} + (x^2 + y^2)^{2n}F^2}.$$

We readily verify that this transformation transforms $x/(x^2+y^2)$ into its nth power.

5. *Third Method. Finite Cremona Transformations.* The variables in (1) are now assumed homogeneous and

$$x_i' = \phi_{i1} \equiv \phi_i(x_1, \cdots, x_m), \qquad (i = 1, \cdots, m),$$

the ϕ's being forms of order n, are assumed to define a Cremona transformation of finite period p. The rth iterate

of the ordered functions $\phi_{11}, \cdots, \phi_{m1}$ is denoted by $\phi_{1r}, \cdots, \phi_{mr}$, so that

$$\phi_{i0}(x_1, \cdots, x_m) \equiv x_i,$$
$$\phi_{ir}(x_1, \cdots, x_m) \equiv \phi_{i,r-1}(\phi_{11}, \cdots, \phi_{m1}).$$

Evidently

$$\phi_{1p} : \phi_{2p} : \cdots : \phi_{mp} = x_1 : x_2 : \cdots : x_m.$$

Since ϕ_{ip} is a form of order n^p, there exists a form $\psi(x_1, \cdots, x_m)$ of order $n^p - 1$ such that

$$x_i \psi(x_1, \cdots, x_m) = \phi_{ip}(x_1, \cdots, x_m).$$

Hence

$$\begin{aligned}\phi_{i1}(x_1, \cdots, x_m)\psi(\phi_{11}, \cdots, \phi_{m1}) &= \phi_{i,p+1}(x_1, \cdots, x_m) \\ &= \phi_{i1}(\phi_{1p}, \cdots, \phi_{mp}) \\ &= \phi_{i1}[x_1\psi(x_1, \cdots, x_m), \cdots, x_m\psi(x_1, \cdots, x_m)] \\ &= \phi_{i1}(x_1, \cdots, x_m)\psi^n(x_1, \cdots, x_m),\end{aligned}$$

since $\phi_{i1}(x_1, \cdots, x_m)$ is a form of order n. We conclude that

$$\psi(\phi_{11}, \cdots, \phi_{m1}) = \psi^n(x_1, \cdots, x_m).$$

Hence $\psi(x_1, \cdots, x_m)$ is a solution of (1). *There exists a solution of* (1) *corresponding to any Cremona transformation of order n and finite period.*

The function ψ thus determined need not be the simplest solution of the functional equation for a given Cremona transformation : it may be factorable, and one of its factors may be a solution. For example, for the Cremona transformation

$$\begin{aligned}x' &= 3xyz - x^2w - 2y^3, \\ y' &= 2xz^2 - xyw - y^2z, \\ z' &= xyz - 2y^2w + yz^2, \\ w' &= xw^2 - 3yzw + 2z^3,\end{aligned}$$

of period 2, the function ψ is found to be

$$(x^2w^2 - 3y^2z^2 + 4xz^3 + 4wy^3 - 6xyzw)^2.$$

The square root of this function is transformed into its cube by the Cremona transformation, and we have Eisenstein's automorphism.

6. *Automorphisms of Determinants.* If the elements of a determinant $D \equiv |a_{ij}|$ of order n are independent variables, and A_{ij} is the cofactor of a_{ij}, the transformation $a'_{ij} = A_{ij}$ is a Cremona transformation of period 2, which transforms D into D^{n-1}.

Now suppose each element is a function of m variables, say

$$a_{ij} \equiv a_{ij}(x_1, \cdots, x_m),$$

and let

$$A_{ij} \equiv A_{ij}(x_1, \cdots, x_m).$$

In general, the equations

$$a_{ij}(x_1', \cdots, x_m') = A_{ij}(x_1, \cdots, x_m)$$

are inconsistent, and, when consistent, do not define $x_1', \cdots, x_m'$ as rational integral functions of $x_1, \cdots, x_m$. In certain special cases we do obtain a transformation of the desired type.

Consider, for example, the symmetric determinant

$$\begin{vmatrix} a & d & e \\ d & b & f \\ e & f & c \end{vmatrix}.$$

The variables d, e, f occur twice, but their respective cofactors are the same in both cases. The determinant is transformed into its square by the transformation $a' = bc - f^2$, etc., each element being replaced by its cofactor. Eisenstein* credits this diophantine automorphism to Lagrange.

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* Journal für Mathematik, vol. 27, p. 105.

ALL POSITIVE INTEGERS ARE SUMS OF VALUES OF A QUADRATIC FUNCTION OF x*

BY L. E. DICKSON

1. *Introduction.* Fermat stated that he was the first to discover the beautiful theorem that every integer $A \geqq 0$ is a sum of $m+2$ *polygonal* numbers

$$p_{m+2}(x) = \tfrac{1}{2}m(x^2 - x) + x \tag{1}$$

of *order* $m+2$ (or $m+2$ sides), where x is an integer $\geqq 0$. The cases $m=1$ and $m=2$ state that every A is a sum of three *triangular* numbers $p_3(x) = \frac{1}{2}x(x+1)$, and also a sum of four squares $p_4(x) = x^2$.

Cauchy† was the first to publish a proof of Fermat's statement and showed that all but four of the polygonal numbers may be taken to be 0 or 1.

In this paper and its sequel we shall give a complete solution of the following more general question.

PROBLEM. Find every quadratic function $f(x)$ which takes integral values $\geqq 0$ for all integers $x \geqq 0$, such that every positive integer A is a sum of l of these values, where l depends on $f(x)$, but not on A.

2. LEMMA 1. *A quadratic function of x is an integer for every integer $x \geqq 0$ if and only if it is of the form*

$$f(x) = \tfrac{1}{2}mx^2 + \tfrac{1}{2}nx + c, \quad m + n \text{ even}, \tag{2}$$

where m, n, and c are integers.

Consider ux^2+vx+c. By its values for $x=0$, 1, and 2, c, $u+v$, and $4u+2v$ are integers. Subtract the double of the

* Presented to the Society, September 9, 1927.

† Oeuvres, (2), vol. 6, pp. 320–353. Pepin gave a modified proof, Atti dei Lincei, vol. 46 (1892–93), pp. 119–131. His proof requires a separate examination when $A < 110m$. For the simpler proof in §5, the limit is $A < 44m+32$.

second from the third. Hence $2u$ is an integer m. Since $\frac{1}{2}m+v$ is an integer, v is half an integer n.

3. *Positive Quadratic Functions Representing* 0 *and* 1. Let (2) be $\geqq 0$ for every integer $x \geqq 0$, whence $m>0$, $c \geqq 0$. Then

$$f(x+1) - f(x) = mx + \tfrac{1}{2}(m+n)$$

increases with x. Hence $f(x)$ does not represent every positive integer A. Thus $l>1$ in our problem, and a sum of two or more values of $f(x)$ must give $A=1$. Hence $f(u)=1$, $f(k)=0$ for certain integers $u \geqq 0$, $k \geqq 0$. We assume that k has its least value. Then

$$f(x) = f(x) - f(k) = \tfrac{1}{2}(x-k)[m(x+k)+n].$$

Since $f(u)=1$,

$$n = s - m(u+k), \quad s = 2/(u-k),$$

where s is an integer. Thus $u-k=\pm 1$ or ± 2, and

$$f(x) = \tfrac{1}{2}(x-k)[m(x-u)+s].$$

If $u-k = \pm 1$,

$$f(x) = \tfrac{1}{2}(x-k)[m(x-k \mp 1) \pm 2] = p_{m+2}(\pm x \mp k).$$

If $u-k=2$,

$f(k+1) = \frac{1}{2}(1-m) \geqq 0$ gives $m=1$, $f(x) = p_3(x-k-1)$.

Finally, if $u-k=-2$, then $k \geqq 2$ and $f(k-1)=\frac{1}{2}(1-m)$ is zero, since it is not negative. But this contradicts the definition of k as least.

THEOREM 1. *The functions derived from* (1) *by replacing* x *by* $x-k$ *or* $k-x$ *are the only quadratic functions of* x *which are integers* $\geqq 0$ *for every integer* $x \geqq 0$, *and which take the values* 0 *and* 1 *for certain integers* $x \geqq 0$.

The values of $p_2(-x)=\frac{1}{2}(x-1)x$ coincide with the triangular numbers. Hence if $m=1$, our problem is the same when $k \geqq 1$ as when $k=0$. This is evidently true also if $m=2$. Without loss of generality, we may then henceforth take $m>2$.

4. *Polynomials with an Excess.* Let $f(x)$ have an integral value $\geqq 0$ for every integer $x \geqq 0$, and let one value be zero. Let $M_s(A)$ denote the maximum sum $\leqq A$ of s values of $f(x)$, and write $E_s(A)$ for $A - M_s(A)$. In case $E_s(A)$ has a finite maximum E_s for all integers $A \geqq 0$, every integer $A \geqq 0$ is a sum of E_s numbers 0 or 1 and s values of $f(x)$. Then E_s is called the s-excess of $f(x)$. We shall drop the subscript 4 from E_4.

Let $\alpha, \beta, \gamma, \delta$ denote the four integral values $\geqq 0$ of x and write

$$a = \Sigma\alpha^2, \qquad b = \Sigma\alpha. \tag{3}$$

Take (2) as $f(x)$ and insert the four values of x. Thus

$$A = \tfrac{1}{2}ma + \tfrac{1}{2}nb + 4c + r, \qquad 0 \leqq r \leqq E, \tag{4}$$

for a suitable integer r. Cauchy proved the following result.

LEMMA 2. *If a and b are positive odd integers such that $b^2 < 4a$ and*

$$b^2 + 2b + 4 > 3a, \tag{5}$$

equations (3) *have solutions $\alpha, \beta, \gamma, \delta$ in integers $\geqq 0$.*

Multiply (5) by m and replace ma by its value from (4). The resulting inequality follows from that obtained by suppressing $6r$. Multiplication by $4m$ now yields the equivalent inequality

$$(2mb + \tau)^2 > U, \quad U = \tau^2 + 4m(6A - 24c - 4m), \tag{6}$$

where $\tau = 2m + 3n$. This inequality holds if

$$b > (U^{1/2} - \tau)/(2m), \quad U \geqq 0. \tag{7}$$

To satisfy $b^2 < 4a$, multiply by m^2 and replace ma by its value from (4). The resulting condition evidently follows from that with r replaced by E, and hence from

$$(mb + 2n)^2 < 4V, \quad V = n^2 + m(2A - 8c - 2E). \tag{8}$$

This inequality holds if

$$b < (2V^{1/2} - \quad)/m, \quad V \geqq 0, \tag{9}$$

and if $mb+2n \geqq 0$. For $n \geqq 0$, the latter evidently holds if $b>0$. For $n<0$, it holds by (7) if

$$4n - \tau + U^{1/2} \geqq 0 \tag{10}$$

and hence if

$$3A \geqq 12c + 2m - 2n - n^2/m \quad (\text{if } n < 0). \tag{11}$$

We desire that $b>0$. By (7), this will be true if $U^{1/2} \geqq \tau$. If $n<0$, this follows from (10). But if $n \geqq 0$, whence $\tau>0$, it holds if and only if $U \geqq \tau^2$, and hence if the quantity in the last parenthesis of (6) is $\geqq 0$:

$$A \geqq 4c + \tfrac{2}{3}m \quad (\text{if } n \geqq 0). \tag{12}$$

There will be at least d positive integers between the limits on b stated in (7) and (9) if

$$4V^{1/2} - U^{1/2} > P, \quad P = 2md - 2m + n. \tag{13}$$

The left member is $\geqq 0$ if

$$16V \geqq U, \tag{14}$$

and then (13) holds* if its square holds and hence if

$$F \equiv (2V + W)^2 - VU > 0, \quad 8W \equiv U - P^2, \quad P \geqq 0. \tag{15}$$

By the *minor conditions* we shall mean $U \geqq 0$, $V \geqq 0$, the inequality (14), and (11) or (12). Since

$$16V - U = 3(2m - n)^2 + 4n^2 + 8m(A - 4c - 4E), \tag{16}$$

it follows that (14) holds if

$$A \geqq 4c + 4E. \tag{17}$$

The latter implies $V \geqq 0$. Evidently $U \geqq 0$ if

$$A \geqq 4c + \tfrac{2}{3}m. \tag{18}$$

Hence the minor conditions all follow from (17) and (18) if $n \geqq 0$, but from these two and (11) if $n<0$. We shall speak of these as the *reduced* minor conditions.

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* Automatically if $P<0$.

5. *Polygonal Numbers.* We take (1) as the function (2). Thus $n=2-m$, $c=0$. By Table I, $E(2m+3)=m-2$. Hence E is not smaller than the value in

THEOREM 2. *For the function* (1), $E_4=m-2$ *if* $m\geqq 3$.

The reduced minor conditions are all satisfied if $A\geqq 4m$. Here (4) is $A=mg+b+r$, $g=\frac{1}{2}(a-b)$. If b takes the odd values β and $\beta+2$, while r takes the values $0, 1, \cdots, m-2$, the values of $b+r$ are $\beta+j (j=0, 1, \cdots, m)$. These, with $j=m$ omitted, form a complete set of residues modulo m. Hence for any A, the preceding equation yields an integral value of g and hence an odd integral value of a.

If there are at least $d=4$ integers between the limits for b, there will exist the desired two odd values for b. Then (6), (8), (13), and (15) give

$$U = 24mA - 15m^2 - 12m + 36, \quad V = 2mA - m^2 + 4,$$

$$P = 5m + 2, \quad W = 3mA - 5m^2 - 4m + 4,$$

$$F = m^2A^2 - 44m^3A - 32m^2A + 34m^4 + 44m^3 - 56m^2 - 48m > 0.$$

Evidently $F>0$ if $A\geqq 44m+32$.

Next, let $A<44m+32$. Then $A<M\equiv 48m+21$. In Table I the entries involving the same multiple of m, together with all intervening integers, will be said to form a *block*. We suppress $29m+10-12$ and $45m+10-13$. Down to M, the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, whence $E(A)\leqq 1$ for every A within a block. For every $A<M$ not within a block, $E(A)\leqq m-2$. This will follow if proved when $A+1$ is the first number of any abridged block. Then A is the sum of $m-2$ and a number t occurring explicitly in the abridged table except as follows. If $A=10m+4$, $26m+12$, $27m+10$, or $28m+12$, then $A=m-3+t$. If $A=6m+3$, $21m+6$, $28m+7$, $30m+11$, $34m+11$, or $42m+12$, then $A=m-4+t$; while, if $m=3$, A is equal to the A for the next smaller m.

TABLE I.

Sums of Four Polygonal Numbers

0–4, m+2–5, $2m$+4–6, $3m$+3–7, $4m$+5–8, $5m$+7–8, $6m$+4–9, $7m$+6–9, $8m$+8–10, $9m$+7–10, $10m$+5–11, $11m$+7–9, 11, $12m$+8–12, $13m$+8–12, $14m$+10–12, $15m$+6–9, 11–13, $16m$+8–13, $17m$+10–13, $18m$+9–14, $19m$+11–14, $20m$+10–14, $21m$+7–13, 15, $22m$+9–15, $23m$+11–15, $24m$+10–16, $25m$+11–13, 15, 16, $26m$+13–16, $27m$+11–16, $28m$+8–11, 13–17, $29m$+10–12, 15–17, $30m$+12–17, $31m$+11–17, $32m$+13–18, $33m$+15–18, $34m$+12–18, $35m$+14–17, $36m$+9–19, $37m$+11–13, 15–19, $38m$+13–15, 17–19, $39m$+12–17, 19, $40m$+14–20, $41m$+16–20, $42m$+13–20, $43m$+14–17, 19, 20, $44m$+16–20, $45m$+10–13, 16–21, $46m$+12–21, $47m$+14–17, 19–21, $48m$+13–21, $49m$+15–21, $50m$+17–22, $51m$+14–17, 19–22, $52m$+16–22, $53m$+18–21, $54m$+17–19, 21, 22, $55m$+11–17, 19–23, $56m$+13–23, $57m$+15–21, 23, $58m$+14–23, $59m$+16, 17, 19–23, $60m$+16–24, $61m$+15–21, 23, 24, $62m$+17–24, $63m$+19–24, $64m$+17–24, $65m$+16–21, 23, 24, $66m$+12–15, 17–25, $67m$+14–16, 19–25, $68m$+16, 17, 19–25, $69m$+15–18, 20, 21, 23–25, $70m$+17–19, 21–25, $71m$+19–25, $72m$+16–26, $73m$+18–21, 23–26, $74m$+20–23, 25, 26, $75m$+19–25, $76m$+17–26, $77m$+19–21, 23–26, $78m$+13–16, 20–27, $79m$+15–17, 20–25, 27, $80m$+17, 18, 22–27, $81m$+16–21, 23–27, $82m$+18–27, $83m$+19–25, 27, $84m$+17–28, $85m$+19–21, 23–28, $86m$+21–28, $87m$+19–25, 27, 28, $88m$+18–22, 24–28, $89m$+20, 21, 23–28, $90m$+20–23, 25–28, $91m$+14–17, 20–25, 27–29, $92m$+16–18, 22–29, $93m$+18–21, 23–29, $94m$+17–29, $95m$+19, 20, 22–25, 27–29, $96m$+21–29, $97m$+18–21, 23–29, $98m$+20–27, 29, 30, $99m$+20–25, 27–30, $100m$+21–30, $101m$+19–21, 23–29, $102m$+21–30, $103m$+22–25, 27–30, $104m$+22–30, $105m$+15–18, 24–29, 31, $106m$+17–23, 25–31, $107m$+19, 20, 22–25, 27–31, $108m$+18–21, 24–31, $109m$+20, 21, 23–29, 31, $110m$+22–31, $111m$+19–25, 27–31, $112m$+21–32, $113m$+23–29, 31, 32, $114m$+22–27, 29–32, $115m$+20–22, 24, 25, 27–32, $116m$+22, 23, 25–32, $117m$+23–29, 31, 32, $118m$+23–32, $119m$+22–25, 27–32, $120m$+16–19, 21–33, $121m$+18–20, 23–29, 31–33, $122m$+20, 21, 25–33, $123m$+19–25, 27–33, $124m$+21, 22, 25–33, $125m$+23, 25–29, 31–33, $126m$+20–31, 33, $127m$+22–25, 27–33, $128m$+24–26, 28–34, $129m$+23–29, 31–34, $130m$+21–23, 25–27, 29–34, $131m$+23, 24, 28–33, $132m$+24–34, $133m$+23–29, 31–34, $134m$+25–34, $135m$+22–24, 27–33, $136m$+17–20, 24–35, $137m$+19–21, 26–29, 31–35, $138m$+21, 22, 25, 26, 28–35, $139m$+20–23, 27–33, 35, $140m$+22, 23, 26, 27, 29–35, $141m$+23–29, 31–35, $142m$+21–31, 33–35, $143m$+23–25, 27–33, 35, $144m$+25–36, $145m$+24–29, 31–36, $146m$+22–27, 29–36, $147m$+24, 25, 27–33, 35, 36, $148m$+24–27, 29–36, $149m$+25–29, 31–36, $150m$+25–36, $151m$+23–25, 27–33, 35, 36, $152m$+25–36, $153m$+18–21, 27–29, 31–37, $154m$+20–22, 26–37, $155m$+22, 23, 28–33, 35–37, $156m$+21–37, $157m$+23–29, 31–37, $158m$+25–30, 33–35, 37, $159m$+22–25, 28–33, 35–37, $160m$+24–37, $161m$+26, 28, 29, 31–37, $162m$+25–27, 29–38, $163m$+23–25, 27–33, 35–38, $164m$+25–27, 30–38, $165m$+26–28, 31–37, $166m$+26–38, $167m$+28–33, 35–38, $168m$+24–26, 29–31, 33–38, $169m$+26–29, 31–37, $170m$+28–38, $171m$+19–22, 27–33, 35–39, $172m$+21–23, 26–39, $173m$+23, 24, 28, 29, 31–37, 39, $174m$+22–31, 33–35, 37–39, $175m$+24, 25, 27–33,

35–39, $176m+26$, 29–39, $177m+23$–26, 28, 29, 31–37, 39, $178m+25$–27, 29–39, $179m+27$, 31–33, 35–39, $180m+26$–40, $181m+24$–29, 31–37, 39, 40, $182m+26$–40, $183m+27$–33, 35–40, $184m+27$–40, $185m+29$, 31–37, 39, 40, $186m+25$–40, $187m+27$–33, 35–40, $188m+29$, 30, 32–40, $189m+27$–29, 31–37, 39, 40, $190m+20$–23, 29–31, 33–35, 37–41, $191m+22$–24, 28–33, 35–41, $192m+24$–41, $193m+23$–26, 28, 29, 31–37, 39–41, $194m+25$, 26, 30–32, 34–39, 41, $195m+27$, 29–33, 35–41, $196m+24$–27, 29–41, $197m+26$–28, 31–37, 39–41, $198m+28$–41, $199m+27$, 28, 30–32, 35, 36.

If $A=15m+5$ or $46m+11$, then $A=m-5+t$; while, if $m=3$ or 4, A is $\leqq$ the A for the next smaller m. Finally, if $A=36m+8$, then $A=m-6+t$; while, if $m=3$, 4, or 5, $A=34m+14$, 16, or 18, which belong to an earlier block. This completes the proof of Theorem 2.

By that theorem, every integer $A\geqq 0$ is a sum of $m+2$ polygonal numbers. Hence $E_s=0$ if $s\geqq m+2$. Next, let $4\leqq s<m+2$. If a sum by s of the polygonal numbers 0, 1, $m+2$, $3m+3$, $\cdots$ is $\leqq 2m+3$, at most one summand is $m+2$, whence the maximum such sum is $m+2+s-1$. Hence $E_s(2m+3)=m-s+2$. By Theorem 2, A is a sum of four polygonal numbers and $m-2$ numbers 0 or 1. Regard $s-4$ of the latter as polygonal numbers. Hence A is a sum of s polygonal numbers and $m-s+2$ numbers 0 or 1. All of these facts prove the following theorem.

THEOREM 3. *For the function* (1), $E_s=0$ *if* $s\geqq m+2$, *while* $E_s=m-s+2$ *if* $4\leqq s\leqq m+2$.

In the second case, $s+E_s=m+2$, so that the use of 5 or more polygonal numbers >1 yields no gain (but rather a loss) over the use of only four.

6. *Deductions from Table* I. We extended our table beyond the limit $48m+21$ required for the proof of Theorem 2 in order to deduce interesting facts concerning $E(A)$ for function (1), which are essential to the sequel.

LEMMA 3. *For* $54m+17\leqq A<74m+28$, $E(A)\leqq m-6$ *if* $m\geqq 7$, $E(A)\leqq 1$ *if* $m=5$ *or* 6.

From Table I we suppress $67m+14$-16. Since the difference between any two consecutive numbers in any abridged block is now $\leqq 2$, $E(A)\leqq 1$, for every A within a

block. Let f be the term free of m in the leader $qm+f$ of any abridged block. First, let $m=5$. For $q=56, \cdots, 74$, we find that $f+4$ is the term free of m in a number of the preceding abridged block. Hence $qm+f-1$ is the sum of $m-5$ and the number $(q-1)m+f+4$ in the abridged table. Finally, the E of $55m+10=53m+20$ is 1. Second, let $m\geqq 6$. Except for $q=55, 56, 62, 70$, $f+5$ is the term free of m in a number of the preceding abridged block, whence $E(A)\leqq m-6$. For the four q's, we use $f+6$ if $m>6$. If $m=6$, $56m+12=55m+18$, $62m+16=61m+22$, $70m+16=69m+22$, which fall within preceding blocks, while $55m+1<54m+17$.

LEMMA 4. *For* $74m+20\leqq A\leqq 199m+37$, $E(A)\leqq 1$ *if* $m=7$, $E(A)\leqq m-7$ *if* $m\geqq 8$, *except that* $E(80m+21)=m-6$ *if* $m=8$ *or* 9.

From each block we suppress all entries down to and including the last entry which differs by 3 or more from the next entry. As in Lemma 3, $f+6$ succeeds except for $q=80$, 106, 156, 158, 169, 195. For $m=7$, $a=80m+21$ equals $79m+28$, whose E is 1. For $m\geqq 10$, we restore the previously excluded $80m+17$-18 and then have the permissible value $E(a)=3$ and $80m+16=m-7+79m+23$. But for $m=8$ or 9, $80m+18\leqq 79m+27=a'$, and the full table includes no number numerically between a' and a, whence $E(a)=m-6$.

For $q=106$, 169, 195, we may use $f+6$ after suppressing $106m+17$-23, $169m+26$-29, and $195m+27$.

For $m\geqq 8$, $b=156m+20=m-8+155m+28$. But if $m=7$, $b=155m+27$, which was treated under $q=155$.

Finally, let $q=158$. If $m\geqq 9$ we restore the previously excluded $158m+25$-30, noting that $E(158m+32)=2$ is a permissible value. Then $158m+24=m-7+t$, where $t=157m+31$ is in the table. If $m=7$ or 8, the latter result is applicable since the missing $158m+31$-32 are equal to two of $159m+23$-25.

Assistance was provided by the Carnegie Institution for the construction of Table I and checking results by it.

THE UNIVERSITY OF CHICAGO

THE SUMMABILITY OF FOURIER SERIES

BY M. H. STONE

In an article of mine which appeared recently in the Transactions of this Society, I applied the general theorems deduced to the particular case of Fourier series. In these applications I omitted an essential step.* The object of this note is to extend some of the methods of that paper in the case of Fourier series, to prove some general theorems about the summability of infinite series, and to supply thereby the reasoning which I omitted in the earlier treatment.

M. Riesz has given a general method of summing an infinite series,† of which a special case may be described as follows: a series of functions of the real variable x,

$$u_0(x) + u_1(x) + u_2(x) + \cdots,$$

is said to be summable (α^Δ, δ) to the value $U(x)$ if

$$S_\Omega^{\Delta,\delta}(x) \to U(x), \quad \Omega \to +\infty,$$

where

$$S_\Omega^{\Delta,\delta} \equiv \sum_{\alpha=0}^{<\omega} \left(1 - \frac{\alpha^\Delta}{\Omega}\right)^\delta u_\alpha(x), \ \Omega = \omega^\Delta, \ \Delta > 0, \ \delta \geqq 0.$$

Here ω is a continuous variable. A more general method of summation is obtained if ω is restricted to a sequence of values $0 < \omega_1 < \omega_2 < \cdots, \omega_\nu \to +\infty$. If

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* Stone, Transactions of this Society, vol. 28 (1926), pp. 695–761. See Theorems 14, 33, 35, 36, 37, 38.

† M. Riesz, Comptes Rendus, vol. 149 (1909), pp. 909–912; Comptes Rendus, vol. 152 (1911), pp. 1651–1654; Proceedings of the London Society, (2), vol. 22 (1923–24), pp. 412–419; Hardy and Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tracts, No. 18, 1915), Chapters IV and V; Hardy, Proceedings of the London Society, (2), vol. 15 (1915–16), pp. 72–88.

$$S_\Omega^{\Delta,\delta}(x) \to U(x),$$

the series may be referred to as summable $(\alpha^\Delta, \delta; \omega_\rho)$ to the value $U(x)$.* Clearly summability by the first method implies summability by the second; but the converse is not true in general.† The omission referred to above is due to the neglect of this fact. In order to remedy it, we may follow one of two courses: we may extend the method involving summability $(\alpha^\Delta, \delta; \omega_\rho)$ which was employed in that paper to yield a method involving summability (α^Δ, δ); or we may prove that with respect to Fourier series the two methods are interchangeable. We shall investigate both possibilities.

It is convenient to introduce the definition of a restricted kind of equivalence.

DEFINITION. *Two definitions of summability are said to be equivalent with respect to a class of series when every series of that class summable by one method is summable by the other to the same sum; and when, further, uniform summability on a range of the variable x in one case implies uniform summability on the same range in the other.*

In examining the two types of summability defined above and their equivalence with respect to the class of Fourier series and certain related classes of series, we shall restrict Δ to integral values. It is first necessary to find a compact analytical expression for the sums $S_\Omega^{n,\delta}$. Without loss of generality we may suppose that δ is greater than zero; and may assume that the Lebesgue-integrable function $f(x)$

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* This type of summability has been applied to the study of Fourier series by W. H. Young, Leipziger Berichte, vol. 63 (1911), pp. 369–387; by Stone, loc. cit.; and by Bailey, Annals of Mathematics, vol. 27 (1926), pp. 69–102.

† Riesz, Comptes Rendus, vol. 152 (1911), pp. 1651–1654; Proceedings of the London Society, vol. 22 (1923–24), pp. 412–419. Bailey, loc. cit., Lemmas III and IV, states two equivalence theorems which in view of Riesz's London paper do not seem to be correct; his reference to Young seems to be in error, as Young, loc. cit., does not give any discussion of equivalence.

whose Fourier series is under consideration satisfies the condition $\int_0^1 f\,dy=0$, since this restriction modifies only the constant term of the Fourier series on (0, 1). If we let $G(x, y; \lambda)$ be the Green's function for the differential system $u'+\lambda u=0$, $u(0)-u(1)=0$, so that*

$$G(x,y;\lambda) = \{e^{\lambda(y-x)};0\} + e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda}),\quad \mathcal{R}(\lambda)\geqq 0,$$

$$G(x,y;\lambda) = \{0;-e^{\lambda(y-x)}\} + e^{\lambda y+\lambda(1-x)}/(1-e^{\lambda}),\ \mathcal{R}(\lambda)\leqq 0,$$

and if we choose appropriate contours S_1 and S_2 enclosing the poles of $\int_0^1 f(y)G(x, y; \lambda)dy$ on the upper and lower half of the λ-plane respectively, we may write

$$S_\Omega^{n,\delta} = \frac{1}{2\pi i}\int_{S_1}\int_0^1 f(y)[1-(-i\lambda/\Lambda)^n]^\delta\left\{\frac{\partial^k G}{\partial x^k};\frac{\partial^k G}{\partial x^k}\right\}dy\,d\lambda$$

$$+\frac{1}{2\pi i}\int_{S_2}\int_0^1 f(y)[1-(+i\lambda/\Lambda)^n]^\delta\left\{\frac{\partial^k G}{\partial x^k};\frac{\partial^k G}{\partial x^k}\right\}dy\,d\lambda,$$

$$\Lambda = 2\pi\omega, \qquad (k=0,1,2,\cdots),$$

for the sum obtained from the kth derived series of the Fourier series on the interval (0, 1) for the function $f(x)$.† The contours S_1 and S_2 which we shall employ may be described as the boundaries of the two semicircular regions into which the circle $|\lambda|\leqq\Lambda$ is divided by the axis of reals. It will be convenient to denote the circular arcs in S_1 and S_2 by C_1 and C_2 respectively. The formula given above can then be verified by the calculus of residues; it is valid even when C_1 and C_2 pass through poles of the Green's function, because of the presence of the factors $(1-(\pm i\lambda/\Lambda)^n)^\delta$, $\delta>0$.

We first study the contribution to $S_\Omega^{n,\delta}$ from the bracketed terms in the Green's function. The method followed is sufficiently exemplified by the treatment of the portion of this contribution arising from that part of S_1 for which

* Stone, loc. cit., p. 717.

† Stone, loc. cit., §2 and Lemma 13.

$\mathcal{R}(\lambda) \geqq 0$. The integral over this path may be replaced by an integral from 0 to Λi along the axis of imaginaries, and therefore becomes

$$\frac{1}{2\pi i}\int_0^{\Lambda i}\int_0^z f(y)[1-(-i\lambda/\Lambda)^n]^\delta(-\lambda)^k e^{\lambda(y-z)}dy\,d\lambda$$

$$=\frac{1}{2\pi}\int_0^z\int_0^1 f(-z+x)(1-\phi^n)^\delta(-i\Lambda\phi)^k\Lambda e^{-\Lambda i\phi z}d\phi\,dz$$

by the change of variables $z=x-y$, $\phi=-\lambda i/\Lambda$. The total contribution from the bracket terms is found in this way to be

$$\frac{1}{2\pi}\int_0^{1-z} f(z+x)\Lambda^{k+1}\Phi_{n,k,\delta}(\Lambda z)dz$$

$$+\frac{(-1)^k}{2\pi}\int_0^z f(-z+x)\Lambda^{k+1}\Phi_{n,k,\delta}(\Lambda z)dz,$$

where

$$\Phi_{n,k,\delta}(\alpha)=\int_0^1(1-\phi^n)^\delta\phi^k[(-i)^k e^{\alpha i\phi}+i^k e^{-\alpha i\phi}]d\phi.$$

By integrating $k+1$ times by parts in the integral defining $\Phi_{n,k,\delta}(\alpha)$ we can obtain a more convenient form for this function when $\delta \geqq k$. We denote by $h_m(\phi)$ the mth derivative of the function $(1-\phi^n)^\delta\,\phi^k$. If δ is greater than or equal to k the functions $h_0, \cdots, h_k$ are continuous functions of ϕ on the interval $(0, 1)$; and h_{k+1} is continuous except at $\phi=1$ and is integrable in the sense of Lebesgue on $(0, 1)$. Furthermore, the equations $h_m(0)=h_m(1)=0$ are satisfied when $m=0, \cdots, k-1$; and $h_k(1)=0$ if $\delta>k$. Thus the result of integrating $k+1$ times by parts is seen to be*

* In a similar integration by parts which occurs in the proof of Theorem 35 of my paper cited above one term has been overlooked; the necessary correction is easily supplied.

$$\Phi_{n,k,\delta}(\alpha) = (-1)^k \alpha^{-k} \int_0^1 h_k(\phi)[(-1)^k e^{\alpha i\phi} + (-1)^k e^{-\alpha i\phi}]d\phi$$

$$= 2\alpha^{-k} \int_0^1 h_k(\phi) \cos \alpha\phi \, d\phi$$

$$= 2\alpha^{-k-1} h_k(1) \sin\alpha - 2\alpha^{-k-1} \int_0^1 h_{k+1}(\phi) \sin \alpha\phi \, d\phi.$$

By the theorem of Riemann-Lebesgue, the integral in the last expression is $o(1)$.

We now apply this result to the further simplification of the contribution from the bracket terms investigated above. We restrict the variable x to lie on a fixed interval (a, b) completely interior to $(0, 1)$. We can then choose a positive number A less than both a and $1-b$. Now

$$\left| \frac{1}{2\pi} \int_A^{1-x} f(z+x) \Lambda^{k+1} \Phi_{n,k,\delta}(\Lambda z) dz - \frac{h_k(1)}{\pi} \int_A^{1-x} f(z+x) z^{-k-1} \sin \Lambda z \, dz \right|$$

is less than or equal to

$$\frac{1}{\pi} \int_A^{1-x} |f(z+x)| \left| z^{-k-1} \int_0^1 h_{k+1}(\phi) \sin \Lambda z\phi \, d\phi \right| dz$$

$$\leqq A^{-k-1}\eta \int_A^{1-x} |f(z+x)| \, dz \leqq A^{-k-1}\eta \int_0^1 |f(y)| \, dy \leqq \epsilon$$

for $\Lambda z \geqq \Lambda A$ sufficiently large. We can treat the second part of the contribution in the same way. Thus we can write the whole expression as

$$\frac{1}{2\pi} \int_0^A [f(z+x) + (-1)^k f(-z+x)] \Lambda^{k+1} \Phi_{n,k,\delta}(\Lambda z) dz + \frac{h_k(1)}{\pi} \left[\int_A^{1-x} f(z+x) z^{-k-1} \sin \Lambda z \, dz + \int_A^x (-1)^k f(-z+x) z^{-k-1} \sin \Lambda z \, dz \right] + o(1).$$

If we change the variables of integration in the second term, it may be written

$$\frac{h_k(1)}{\pi}\int_0^1 f(y)K(x,y\,;\,\Lambda)dy$$

where

$$K(x,y\,;\,\Lambda) = \sin\Lambda(y-x)/(y-x)^{k+1}, \quad 0 \leqq y \leqq x-A,$$
$$K(x,y\,;\,\Lambda) = 0 \qquad\qquad x-A < y < x+A,$$
$$K(x,y\,;\,\Lambda) = \sin\Lambda(y-x)/(y-x)^{k+1}, \quad x+A \leqq y \leqq 1,$$

for $a \leqq x \leqq b$; and this integral can easily be shown to be $o(1)$ on (a, b) by the theory of singular integrals.* The final form of the contribution from the bracket terms in G is therefore

$$\frac{1}{2\pi}\int_0^A [f(z+x)+(-1)^k f(-z+x)]\Lambda^{k+1}\,\Phi_{n,k,\delta}(\Lambda z)dz + o(1),$$

uniformly on (a, b), when δ is a positive number greater than or equal to k.

We now turn to the study of the contribution to $S_\Omega^{n,\delta}$ from the remainder of the Green's function, under the assumption that x lies on (a, b) and that δ is a positive number greater than or equal to k. First we show that the integrals taken along the real axis are negligible, and then discuss the integrals over C_1 and C_2.

The contribution from the positive real axis is seen to be

$$\frac{1}{2\pi i}\int_0^\Lambda \{[1-(-i\lambda/\Lambda)^n]^\delta - [1-(i\lambda/\Lambda)^n]^\delta\} \times \int_0^1 [f(y)(-\lambda)^k e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda})]dy\,d\lambda.$$

Under our assumption that $\int_0^1 f\,dy$ vanishes the integrand is analytic in λ at the origin. Furthermore

* Lebesgue, Annales de Toulouse, (3), vol. 1 (1909), pp. 52–55.

$$\int_0^1 [f(y)(-\lambda)^k e^{\lambda(y-1)-\lambda x}/(1-e^{-\lambda})]\,dy = O(\lambda^k e^{-\lambda x}) = O(\lambda^k e^{-\lambda a}).$$

If we split the integral over $(0, \Lambda)$ into two others, one over $(0, \Lambda^{1/2})$, the other over $(\Lambda^{1/2}, \Lambda)$, we find the first to be

$$O\left(\int_0^{\Lambda^{1/2}} O(\Lambda^{-n/2})O(\lambda^k e^{-\lambda a})d\lambda\right) = o(1)$$

and the second to be

$$O\left(\int_{\Lambda^{1/2}}^{\Lambda} O(1)O(\lambda^k e^{-\lambda a})d\lambda\right) = o(1).$$

In exactly the same way we can show that the integral over the negative axis of reals is also $o(1)$ uniformly on (a, b).

In discussing the integrals over the arcs C_1 and C_2 we shall give the details of the method used only for the integral taken over the arc c of C_1 lying on the first quadrant. This may be written

$$\int_0^1 f(y)k(x, y\,;\,\Lambda)dy,$$

where

$$k(x, y\,;\,\Lambda) = \frac{1}{2\pi i}\int_c \frac{[1-(-i\lambda/\Lambda)^n]^\delta(-\lambda)^k e^{\lambda(y-1)-\lambda x}}{(1-e^{-\lambda})}d\lambda.$$

It is our purpose to show that

$$k(x, y\,;\,\Lambda) = O(1),$$

$$\int_\alpha^\beta k(x, y\,;\,\Lambda)\,dy = o(1), \quad 0 \leqq \alpha < \beta \leqq 1,$$

uniformly on (a, b); the theory of singular integrals then shows that $\int_0^1 f(y)\,k(x, y; \Lambda)dy$ is $o(1)$ uniformly on (a, b).* We introduce the variable θ defined by the equation $e^{-i\theta} = -i\lambda/\Lambda$. When λ is on c, θ is a real variable on the interval $(0, \pi/2)$, so that

* Lebesgue, loc. cit.

$$[1-(-i\lambda/\Lambda)^n]^\delta = \theta^\delta O(1),$$

$$(-\lambda)^k e^{\lambda(y-1)-\lambda x} = O(\Lambda^k e^{-\Lambda\Lambda\theta/2}),$$

$$\int_\alpha^\beta (-\lambda)^k e^{\lambda(y-1)-\lambda x} dy = O(\Lambda^{k-1} e^{-\Lambda\Lambda\theta/2}),$$

$$d\lambda = \Lambda O(1) d\theta.$$

For a given value of Λ, we determine an integer l such that $(2l-1)\pi \leqq \Lambda \leqq (2l+1)\pi$. If we then set $\lambda = \mu + 2l\pi i$ we find

$$1/(1-e^{-\lambda}) = 1/(1-e^{-\mu}) = O(1+1/|\mu|)$$

in the region composed of the half-plane $\Re(\mu) \geqq m > 0$ and the rectangle $0 \leqq \Re(\mu) \leqq m$, $-3\pi/2 \leqq \Im(\mu) \leqq +3\pi/2$. If Λ and l are large c lies on the corresponding region for the variable λ. In terms of the variable θ we have

$$1/|\mu| = (1/\Lambda)|e^{-i\theta} - 2l\pi/\Lambda| = (1/\Lambda)[4l\pi(1-\cos\theta)/\Lambda + (1-2l\pi/\Lambda)^2]^{1/2}$$

$$= O\{(1/\Lambda)[4l\pi(1-\cos\theta)/\Lambda]^{1/2}\} = O\{1/(\Lambda\theta)\},$$

when λ is on c. Consequently

$$1/(1-e^{-\lambda}) = O\{1/(\Lambda\theta)\} + O(1).$$

Thus

$$k(x,y;\Lambda) = O\left(\int_0^{\pi/2} \Lambda^k \theta^{\delta-1} e^{-\Lambda\Lambda\theta/2} d\theta\right) + O\left(\int_0^{\pi/2} \Lambda^{k+1} \theta^\delta e^{-\Lambda\Lambda\theta/2} d\theta\right)$$

$$= O(1),$$

if δ is a positive number not less than k; and

$$\int_\alpha^\beta k(x,y;\Lambda) dy = O\left(\int_0^{\pi/2} \Lambda^{k-1} \theta^{\delta-1} e^{-\Lambda\Lambda\theta/2} d\theta\right) + O\left(\int_0^{\pi/2} \Lambda^k \theta^\delta e^{-\Lambda\Lambda\theta/2} d\theta\right)$$

$$= O(1/\Lambda)$$

uniformly on (a, b) under the same conditions.* This establishes the desired result. It is evident now that the integrals over the other parts of C_1 and C_2 will yield to similar treatment. Thus we can state the first theorem.

THEOREM I. *The sum $S_\Omega^{n,\delta}$ formed for the kth derived series of the Fourier series for a Lebesgue-integrable function $f(x)$ on the interval $(0, 1)$ can be expressed as*

$$S_\Omega^{n,\delta} = \frac{1}{2\pi}\int_0^A [f(z+x) + (-1)^k f(-z+x)]\Lambda^{k+1} \cdot \Phi_{\delta,k,n}(\Lambda z)dz + o(1),$$

uniformly $0 < a \leqq x \leqq b < 1$, where δ is a positive number not less than k and A is a positive number less than both a and $1-b$. Here $\Lambda = 2\pi\omega$ is a continuous variable.

This theorem amounts to an extension of the methods of our paper cited above to include a treatment of summability (α^n, δ); it serves to bridge the gap which was left in the proofs of those theorems enumerated in the first footnote. By using this theorem we can arrive at further theorems, concerning the relation of summability (α^n, δ) to summability $(\alpha^n, \delta; \omega_\nu)$ with respect to the class of Fourier series and the class of derived series of Fourier series.

If $f(x)$ is absolutely continuous—that is, if it is an integral in the sense of Lebesgue—we can integrate by parts in the integral appearing in Theorem I. In view of the fact that

$$\int \Phi_{n,k,\delta}(\Lambda z)dz = -\Phi_{n,k-1,\delta}(\Lambda z)/\Lambda, \qquad k \geqq 1,$$

we obtain for $k \geqq 1$

$$S_\Omega^{n,\delta} = -\Lambda^k[f(A+x) + (-1)^k f(-A+x)]\Phi_{n,k-1,\delta}(\Lambda A) + \frac{1}{2\pi}\int_0^A [f'(z+x) + (-1)^{k-1} f(-z+x)]\Lambda^k \Phi_{n,k-1,\delta}(\Lambda z)dz + o(1).$$

* See Stone, loc. cit., Lemma V, for a more detailed discussion of the final steps here.

To evaluate the first term here we treat $\Phi_{n,k-1,\delta}(\Lambda A)$ by the method of integration by parts outlined above. Since $\delta \geqq k$, we have

$$\Phi_{n,k-1,\delta}(\alpha) = -2\alpha^{-k}\int_0^1 \frac{d^k}{d\phi^k}[(1-\phi^n)^\delta \phi^{k-1}]\sin \alpha\phi\, d\phi.$$

In consequence the first term is $o(1)$ uniformly $a \leqq x \leqq b$. By repeating this process we obtain the second theorem.

THEOREM II. *If $f^{(l-1)}(x)$ is absolutely continuous,*

$$S_\Omega^{n,\delta} = \frac{1}{2\pi}\int_0^A [f^{(l)}(z+x) + (-1)^{(k-l)} f^{(l)}(-z+x)]\Lambda^{k-l+1} \cdot \Phi_{n,k-l,\delta}(\Lambda z)dz + o(1),$$

uniformly on (a, b), if $\delta \geqq k \geqq 1$ and $l \leqq k$. In other words, the sum formed for the kth derived series for $f(x)$ and the sum formed for the $(k-l)$th derived series for $f^{(l)}(x)$ have a difference which is uniformly $o(1)$ on (a, b), $l = 0, \cdots, k$.

We now compare summability (α^n, δ) and $(\alpha^n, \delta; \omega_\nu)$ for a Fourier series, $\delta > 0$, under the hypothesis that $\omega_{\nu+1} - \omega_\nu = o(1)$. A necessary and sufficient condition that the series be summable $(\alpha^n, \delta; \omega_\nu)$ to the sum $S(x)$ is that

$$I_\nu = \frac{1}{2\pi}\int_0^A F(z)\Lambda_\nu \Phi_{n,0,\delta}(\Lambda_\nu z)dz = o(1)$$

$$F(z) = f(z+x) + f(-z+x) - 2S(x), \quad \Lambda_\nu = 2\pi\omega_\nu,$$

the limit being approached uniformly on any closed set $\mathcal{E}$ on which the summability is uniform.* We consider the quantity

$$J_\nu = \frac{1}{2\pi}\int_0^A F(z)[\Lambda\Phi_{n,0,\delta}(\Lambda z) - \Lambda_\nu \Phi_{n,0,\delta}(\Lambda_\nu z)]dz$$

* For amplification of the reasoning here, see Stone, loc. cit., Theorem 14.

where $\Lambda_\nu \leqq \Lambda \leqq \Lambda_{\nu+1}$. By the process of integration by parts used above

$$\Lambda\Phi_{n,0,\delta}(\Lambda z) - \Lambda_\nu\Phi_{n,0,\delta}(\Lambda_\nu z) = \int_0^1 \left\{ \frac{d}{d\phi}(1-\phi^n)^\delta \cdot [\sin \Lambda_\nu \phi z - \sin \Lambda \phi z]/z \right\} dz = O(1),$$

since

$$(\sin \Lambda_\nu \phi z - \sin \Lambda \phi z)/z = \phi \cos \Lambda'_\nu \phi z (\Lambda_\nu - \Lambda),$$

$\Lambda_\nu < \Lambda'_\nu < \Lambda$ by the law of the mean. Consequently,

$$|J_\nu| \leqq K \int_0^A |F(z)|\,dz \leqq K\left(\int_0^A |f(z+x)|\,dz + \int_0^A |f(-z+x)|\,dz + A|S(x)| \right),$$

where K is a suitably chosen positive constant. The last expression is $o(1)$ for each value of x when $A \to 0$. The first two terms are uniformly $o(1)$ on (a, b) by a fundamental property of the Lebesgue integral. On any closed set $\mathcal{E}$ on which the given Fourier series is uniformly summable $(\alpha^n, \delta; \omega_\nu)$, the function $S(x)$ is continuous and bounded; on this set $\mathcal{E}$, therefore, $A|S(x)|$ is uniformly $o(1)$ when $A \to 0$. Thus by choosing A sufficiently small we can make $|J_\nu| < \epsilon/2$, where ϵ is a preassigned positive constant. The value assigned to A may be taken independent of x, when x lies on a closed set $\mathcal{E}$ on which the summability $(\alpha^n, \delta; \omega_\nu)$ is uniform. Now

$$S_\Omega^{n,\delta} - S(x) = I_\nu + J_\nu + o(1),$$

where $I_\nu = o(1)$, for any value of x at which $S(x)$ exists. When, for such a value of x, a value of A has been determined so that $|J_\nu| < \epsilon/2$, a number L can be found so that for $\Lambda_{\nu+1} \geqq \Lambda \geqq \Lambda_\nu \geqq L$ we have

$$|I_\nu + o(1)| < \epsilon/2.$$

Thus we find

$$S_\Omega^{n,\delta} - S(x) = o(1),$$

the limit being uniform on any closed set $\mathcal{E}$ on which the summability $(\alpha^n, \delta; \omega_\nu)$ is uniform. Thus we can assert that summability $(\alpha^n, \delta; \omega_\nu)$ and summability (α^n, δ) are equivalent on (a, b) with respect to the class of Fourier series.

In order to extend the result about equivalence to the whole interval (0,1), we need only appeal to an elementary property of Fourier series. If $f(x)$ is defined outside the interval (0, 1) by the periodic relation $f(x+1)=f(x)$, it is found that the Fourier series for $f(x)$ and $f(x-1/2)$ have a simple and important relationship: when the variable x in the second series is replaced by $x+1/2$, the first series results. Thus the behavior of the Fourier series for $f(x)$ at $x=0$ or $x=1$ can be completely discussed by considering the behavior of the Fourier series for $f(x-1/2)$ at $x=1/2$. It is thus seen that summability $(\alpha^n, \delta; \omega_\nu)$ and summability (α^n, δ) are equivalent with respect to the class of Fourier series, without restriction as to the range considered.

THEOREM III. *If* $\omega_{\nu+1}-\omega_\nu=o(1)$, *then summability* $(\alpha^n, \delta; \omega_\nu)$ *and summability* (α^n, δ) *are equivalent with respect to the class of Fourier series.*

By combining Theorems II and III we find without difficulty a more general result.

THEOREM IV. *If* $\omega_{\nu+1}-\omega_\nu=o(1)$, *then summability* (α^n, δ) *and summability* $(\alpha^n, \delta; \omega_\nu)$ *are equivalent with respect to the class of kth derived series of the Fourier series of functions whose* $(k-1)$*th derivatives are absolutely continuous,* $\delta \geqq k \geqq 1$.

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ON THE SEPARATION OF THE PLANE BY IRREDUCIBLE CONTINUA†

BY W. A. WILSON

1. *Introduction.* This question was investigated by A. Rosenthal‡ in 1919. His principal results may be stated as follows. If F is the union of two bounded continua, C_1 and C_2, which are irreducible between the points a and b and have no other common points, then the complement of F with respect to its plane consists of two principal component regions (*Hauptgebiete*), each of which has F as its frontier, and possibly of a number of secondary component regions (*Nebengebiete*), each of which has its frontier wholly in either C_1 or C_2.

It is a simple matter to construct two irreducible continua which together constitute the frontier of precisely two of the complementary regions, but which have more than two points in common. For example, let bmc and bnc be two complementary arcs of a circumference, and let am^* and an^* be two wavy lines intersecting only at a and approaching asymptotically bmc and bnc, respectively. Then $C_1=am^*+bmc$ and $C_2=an^*+bnc$ are both irreducible between a and b and divide the plane into three regions, of which two only have C_1+C_2 as their frontier. But C_1 and C_2 have in common three points, not two.

In this particular case, to be sure, Rosenthal's theorem can be used, for C_1+C_2 can be expressed as the union of two continua, irreducible between a pair of points and intersecting only in these points. This is possible here because C_1 and C_2 can each be separated into two irreducible continua having only one common point. Such a decomposability,

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‡ (A) A. Rosenthal, *Teilung der Ebene durch irreduzibele Kontinua*, Sitzungsberichte der Münchener Akademie, 1919, pp. 91–109.

however, is not a general property of irreducible continua, even when no indecomposable continua are involved. Therefore a removal of the requirement that the two irreducible continua have but two points in common is a real extension of the theorem quoted above.

It is the purpose of the present article to give some general conditions under which this extension is possible and also to investigate the frontiers of the secondary regions. The principal results will be found in §§ 5 and 8.

In certain parts of the work free use is made of the oscillatory set of a continuum about a point and its properties, which have been developed by the author in the papers listed below.* The notation of these papers is also used.

2. *Some Generalizations.* For what follows it is convenient to extend the notion of a continuum irreducible between two points in two ways which must be kept distinct.

DEFINITION I. *If A is a closed sub-set of the continuum C, then C is irreducible about A if no proper sub-continuum of C contains A.*

DEFINITION II. *If α and β are closed sets without common points and the continuum C contains one or more points of both sets, then C is irreducible between α and β if no proper sub-continuum of C contains points of both sets.*

The second definition is due to Miss Anna M. Mullikin.† It is readily seen that if A or $\alpha+\beta$ consists of two points, both definitions yield the ordinary continuum irreducible between two points. The following properties will be used.

LEMMA I. *If C is a bounded continuum irreducible about the sum of the two continua A and B, which have no common points, then $C-(A+B)$ is connected.*

* (B) *On the oscillation of a continuum*, Transactions of this Society, vol. 27.

(C) *Some properties of the irreducible continuum*, ibid., vol. 28.

(D) *On the structure of the irreducible continuum*, American Journal, vol. 48.

† (E) Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24.

The proof given by Rosenthal (loc. cit., p. 104) for the case where A and B are points is immediately applicable.

LEMMA II. *Let C be bounded and closed and contain points of both the closed sets α and β. Let $\alpha \cdot \beta = 0$. Then C can be separated into two closed sets C_1 and C_2, such that $C_1 \cdot C_2 = \alpha \cdot C_2 = \beta \cdot C_1 = 0$, or else C contains a sub-continuum irreducible between α and β.*

This theorem is stated and proved in an equivalent form by Miss Mullikin (loc. cit., p. 147).

LEMMA III. *Let C be a bounded continuum irreducible between the closed subsets α and β, which have no common points. Let A be the oscillatory set of C about some point a of α. Then A contains α and, if D is a sub-continuum of C containing a point x of $C-A$ and a point a' of α, D contains A.*

PROOF. We omit the trivial case that C is indecomposable. If A is complete, $C-A$ is a semi-continuum by a theorem proved elsewhere.* Obviously it contains β and hence can contain no point of α. Let E be a sub-continuum of $C-A$ joining x and β. Then $D+E$ joins α and β, and hence equals C. As $A \cdot E = 0$, D contains A.

If A is not complete, it is indecomposable and is not a continuum of condensation.† Then $\overline{C-A}$ is a proper sub-continuum‡ of C and contains β. Hence it contains no point of α. Thus A contains α in this case too. As x is a point of $C-A$, $D+\overline{C-A}=C$; hence D contains $\overline{C-\overline{C-A}}$. Since $\overline{C-\overline{C-A}}$ is a sub-continuum of A and contains all points of A whose distance from a is less than some positive δ, it is identical with A. Thus D contains A.

3. *A Special Type of Frontier Set.* Let $F=H_1+H_2$, where H_1 and H_2 are bounded continua and $F-H_2=H_1$

* See reference (C), p. 543.

† See reference (D), p. 155.

‡ (F) C. Kuratowski, *Théorie des continus irréductibles*, Fundamenta Mathematicae, vol. 3, Theorem 3.

$-H_1 \cdot H_2$ and $F-H_1=H_2-H_1 \cdot H_2$ are connected sets. Certain properties of this type of set and its complementary regions are easily deduced.

If Z denotes the plane, it follows from the fact that $F-H_1$ is connected that $F-H_1$ lies in some one component G_1 of $Z-H_1$. Likewise, $F-H_2$ lies in one component G_2 of $Z-H_2$.

Now let G' be any other component of $Z-H_1$ and G'' any other component of $Z-H_2$. We first show that, if $G' \cdot G_2 \neq 0$, then G_2 contains G'. For otherwise G' would contain points of H_2, the frontier of G_2, since G' is connected. This is impossible, for G_1 is the only component of $Z-H_1$ containing points of H_2. Likewise, either $G'' \cdot G_1=0$ or G_1 contains G''.

A second property regarding these components is that either $G' \cdot G''=0$ or else $G'=G''$ and the frontier of G' is a part of some component of $H_1 \cdot H_2$. To prove this let us assume that $G' \cdot G'' \neq 0$. Then, if G'' contained points not in G', it would contain points of the frontier of G', which is a part of H_1. This is impossible by the second paragraph above. Likewise, G' contains no point not in G'' and hence $G'=G''$. Let g' denote the frontier of G'. Then $g' \subseteq H_1H_2$. As g' is a continuum, it is a part of some component of H_1H_2.

With these preliminaries, we are in a position to prove the following lemma.

4. LEMMA IV. *Let F be the union of two bounded continua H_1 and H_2 having these properties: $F-H_1$ and $F-H_2$ are connected;* $H_1 \cdot H_2=A+B$, where A and B are continua and $A \cdot B=0$; H_1 and H_2 contain sub-continua C_1 and C_2, respectively, such that $\alpha=C_1 \cdot C_2 \cdot A \neq 0$, $\beta=C_1 \cdot C_2 \cdot B$, C_1 and C_2 are irreducible between α and β, and $F=C_1+C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.*

* In view of §2, Lemma I, this requirement may be replaced by the hypothesis that H_1 and H_2 are irreducible about A and B.

PROOF.* It was shown in §3 that, if Z denotes the plane, one component G_1 of $Z-H_1$ contains $H_2-(A+B)$ and one component G_2 of $Z-H_2$ contains $H_1-(A+B)$. Let $K_1=Z-G_1$ and $K_2=Z-G_2$. Then K_1 and K_2 are continua which do not cut the plane.

Now consider $K_1 \cdot K_2$. As K_1 consists of H_1 and the components G' of $Z-H_1$ differing from G_1, and K_2 is similarly constituted, it is evident from §3 that $K_1 \cdot K_2$ consists of $A+B$ and such components G' of $Z-H_1$ as coincide with components G'' of $Z-H_2$. In §3 it was shown that, if $G'=G''$, then g', the frontier of G', is a part of either A or B. In consequence of these facts $K_1 \cdot K_2$ is the sum of two continua, which may be denoted by A' and B', where A' contains A and may contain one or more regions whose frontiers form a part of A, B' contains B and may contain one or more regions whose frontiers form a part of B, and $A' \cdot B'=0$.

Two cases must now be considered: (a) one of the regions G_1 and G_2 is unbounded; (b) neither G_1 nor G_2 is unbounded. We shall complete the proof for the first case and then return to the second.

If G_1 is unbounded, K_1 is bounded. For otherwise a point in K_1 could be joined to one in G_1 by a broken line not cutting the bounded set H_1, which is the frontier of G_1. Likewise K_2 is bounded, if G_2 is unbounded. Thus either K_1 or K_2 is bounded. We have seen that neither of these continua separates the plane and that their divisor is two continua without common points. We can therefore apply a theorem of Miss Mullikin,† which shows that K_1+K_2 cuts the plane into exactly two regions, R_1 and R_2. Let F_1 and F_2 be the frontiers of these regions. If we can show that $F_1=F_2=F$, the theorem is proved. To this end we first observe that, since no point of F_1 or F_2 can be an inner point of K_1 or K_2, then F_1 and F_2 are parts of $H_1+H_2=F=C_1+C_2$.

* (G) This proof is a modification of the proof of Rosenthal's theorem given by S. Straszewicz in his paper *Über die Zerschneidung der Ebene durch abgeschlossene Mengen*, Fundamenta Mathematicae, vol. 7, p. 187.

† See reference (E), p. 160; also reference (G), §§18, 19.

It is easily seen that $R_1+R_2=G_1 \cdot G_2$. Now if m is a point of R_1 and n is one of R_2, there is a broken line in G_1 joining m and n and not cutting H_1, for $G_1 \cdot H_1=0$. Thus H_1 (and in like fashion H_2) is not an $S(m, n)$.*

There are two sub-cases: either F_1 contains C_1 or C_2, or it does not. To fix the ideas, let $C_1 \subset F_1$. Then $F_1 \cdot C_2$ is not void by the previous paragraph. Since C_2 is irreducible between α and β, it follows from §2, Lemma II, that, if $F_1 \cdot C_2 \neq C_2$, then $F_1 \cdot C_2$ is the sum of two closed sets M and N, such that $M \cdot N=M \cdot \beta=N \cdot \alpha=0$. (One of these, but not both, may be void.) Then $F_1=M+C_1+N$. Since $M+A \subseteq H_2$ and $A+C_1 \subseteq H_1$, neither $M+A$ nor $A+C_1$ is an $S(m, n)$. Hence by a theorem of Janiszewski† $M+A+C_1$, and consequently $M+C_1$, is not an $S(m, n)$. Likewise, C_1+N is not an $S(m, n)$ and a second application of the theorem referred to gives the contradiction that F_1 is not an $S(m, n)$. Hence $F_1 \cdot C_2=C_2$, and $F_1=C_1+C_2=F$.

Now suppose that neither C_1 nor C_2 is a part of F_1. Since C_1 is irreducible between α and β, $F_1 \cdot C_1$ is the sum of two closed sets M and N, such that $M \cdot N=N \cdot \alpha=M \cdot \beta=0$. Either M or N, but not both, may be void. For the same reason $F_1 \cdot C_2=P+Q$, where P and Q are closed and $P \cdot Q=P \cdot \beta=Q \cdot \alpha=0$. Then $F_1=(M+P)+(N+Q)$ and $(M+P)(N+Q)=0$, which is a contradiction, as F_1 is a continuum, unless both M and P, or N and Q, are void. If $M=P=0$, $F_1=N+Q \subseteq N+B+Q$. As neither $N+B$ nor $B+Q$ is an $S(m, n)$, this is a contradiction.

Thus in both sub-cases $F_1=C_1+C_2=F$, and in like fashion $F_2=F$.

It now remains to take care of the situation that arises when neither G_1 nor G_2 is unbounded. If v is a point of G_1

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* A set H is an $S(m, n)$ if every continuum joining m and n cuts H.

† (H) Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A: "If P and Q are bounded closed sets, if $P \cdot Q$ is connected, and neither P nor Q is an $S(m, n)$, then $P+Q$ is not an $S(m, n)$." See also Straszewicz, Fundamenta Mathematicae, vol. 4, p. 129.

not on H_2 and the plane is inverted with respect to ν as a center, the image F^* of F will satisfy the conditions of the first case. This follows from the facts that the correspondence between F and F^* is homeomorphic, that the image G_1^* of G_1 is unbounded, that the images of the frontiers of the component regions of $Z-F$ are the frontiers of the images of these regions, and that the property of irreducibility is an invariant of analysis situs. Then by the proof given above there are precisely two components, R_1^* and R_2^*, of $Z-F^*$ which have F^* as their frontier. Inverting again. we have the inverse images R_1 and R_2 of R_1^* and R_2^*, respectively, as the only components of $Z-F$ which have F as a frontier.

5. *A General Theorem.* It is obvious that Rosenthal's theorem is the special case of the above lemma obtained when A and B are points, in which case $H_1=C_1$ and $H_2=C_2$. We can, however, obtain from this lemma the following theorem, which is, aside from one exceptional case, more general than the theorem in question.

THEOREM I. *Let F be the union of two bounded continua C_1 and C_2 having these properties: $C_1 \cdot C_2=\alpha+\beta$, where α and β are closed and $\alpha \cdot \beta=0$; both C_1 and C_2 are irreducible between α and β; and either C_1 and C_2 are both decomposable, or C_1 is indecomposable and C_2 is decomposable and not the union of two indecomposable continua. Then F cuts the plane and is the frontier of exactly two components of its complement.*

PROOF. This theorem is a corollary of §4. If we let A_1 and A_2 denote the oscillatory sets of C_1 and C_2, respectively, about some point a of α, and let B_1 and B_2 have the same meaning for a point b of β, it follows from §2, Lemma III, that $\alpha=A_1 \cdot A_2$ and $\beta=B_1 \cdot B_2$.

If both continua are decomposable, take $A=A_2$ and $B=B_1$, and set $H_1=A+C_1+B=A_2+C_1$ and $H_2=A+C_2+B=C_2+B_1$. Then $H_1 \cdot H_2=A+B=A_2+B_1$. If C_1 is indecomposable, take $A=A_2$ and $B=B_2$, and set $H_1=A+C_1+B=A_2+C_1+B_2$ and $H_2=A+C_2+B=C_2$. Again $H_1 \cdot H_2=A+B$.

In both cases $A \cdot B = 0$, H_1 and H_2 are irreducible about $A+B$, and $F = H_1+H_2 = C_1+C_2$. Hence §4 gives the theorem.

6. *The Secondary Regions.* Let R_1 and R_2 be the two principal regions, which have the frontier $F = C_1+C_2$, and let R' denote any one of the other components of $Z-F$, that is, one of the secondary regions defined by F if any exist.

Since R' is a component of $Z-F$, it is a part of a component of $Z-H_1$ and of a component of $Z-H_2$. It is not a part of both G_1 and G_2, for, as seen in §4, $G_1 \cdot G_2 = R_1+R_2$. To fix the ideas let $R' \subseteq G'$, where G' is a component of $Z-H_1$ different from G_1. By definition of G' and G_1, $G' \cdot H_2 = 0$; hence $\overline{G}' \cdot F \subseteq H_1$. Then, if F' is the frontier of R', $F' \subseteq H_1$. Likewise, if R' is a part of some component of $Z-H_2$ different from G_2, $F' \subseteq H_2$.

Since $F' \subset F$ and $R_1 \cdot R' = 0$ by hypothesis, it follows that, if m is a point of R' and n one of R_1, then F' is an $S(m, n)$. It is also an irreducible $S(m, n)$.* For, if K is a closed sub-set of F' and x is a point of $F'-K$, then for a sufficiently small positive δ there is a circular region U_δ containing x and points of both R' and R_1, but no point of K. Then m and n can be joined by a broken line not cutting K. Thus no proper closed part of F' is an $S(m, n)$.

If $F' \subseteq H_1$, but F' is not a part of C_1, there are two cases to consider. If C_1 is decomposable, it follows from the definition of H_1 (§5) that F' contains points of A_2 not in C_1. If $F' \cdot (C_1-A_1) \neq 0$, we have a contradiction. For then by §2, Lemmas II and III, $F' \supset A_1$. Then $F' \cdot C_1$ and $F' \cdot (A_1+A_2)$ have in common the continuum A_1. As both of these sets are proper closed parts of F', neither is an $S(m, n)$, which gives the contradiction that F' is not an $S(m, n)$. Thus we have $F' \cdot (C_1-A_1) = 0$ and in this case $F' \subseteq A_1+A_2$.

If C_1 is indecomposable, then C_2 is decomposable and not the sum of two indecomposable continua. Then $A_2 \cdot B_2 = 0$ and the argument used in the previous paragraph shows

* A set P is called an irreducible $S(m, n)$ if no proper closed part of P sian $S(m, n)$.

that we arrive at a contradiction if we assume that F' contains points of both A_2 and B_2 not in C_1. As C_1 is indecomposable, $A_1=B_1=C_1$; hence either $F'\subseteq A_1+A_2$, or $F'\subseteq B_1+B_2$.

The same sort of discussion is employed if $F'\subseteq H_2$. Thus we have found that, *if R' is a secondary region defined by the continuum F of §5, then the frontier of R' is a part either of C_1, or of C_2, or of A_1+A_2, or of B_1+B_2.* For a closer determination of this frontier we need the following theorem.

7. THEOREM II. *Let C be a bounded plane continuum irreducible between the points a and b. Let m and n be two points not on C and let K be a closed sub-set of C which is an irreducible $S(m, n)$. Then K is a part or the whole of the oscillatory set of C about one of its points**.

PROOF. CASE I. *K is a continuum of condensation.* Let K_a and K_b be the saturated semi-continua of $C-K$ containing a and b, respectively, and let neither be void. Since K is a continuum of condensation, $C=\overline{K}_a+\overline{K}_b$. Then $K\cdot\overline{K}_a$ and $K\cdot\overline{K}_b$ have a common point c, which is a limiting point of both K_a and K_b. Then the reasoning in §13 of the paper mentioned in reference (C) shows that K is a part of the oscillatory set about c. Similar reasoning establishes the theorem for the case that either K_a or K_b is void.

CASE II. *K is not a continuum of condensation.* Let us assume that K contains neither a nor b. Then there are subcontinua A_k and B_k, irreducible between K and a and b, respectively, and $A_k\cdot B_k=0$.* Then $L=\overline{C-(A_k+B_k)}$ is irreducible‡ between A_k and B_k, and $L\subseteq K$. We first show that L is indecomposable or is the union of two indecomposable continua. For, if not, there is an irreducible de-

* If every oscillatory set of C is complete, the theorem can be deduced easily from Theorem 9 of a paper by R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27 (1925), pp. 416–428.

† See reference (F), Theorems II and IV.

‡ See reference (C), §4.

composition* of L into two proper sub-continua M and N such that

$$M \cdot B_k = N \cdot A_k = 0$$

and either M or N, say N, is decomposable. Obviously N is irreducible between A_k+M and B_k and there is an irreducible decomposition of N into two proper sub-continua P and Q such that

$$P \cdot B_k = (A_k+M) \cdot Q = 0.$$

Then we have

$$C = (A_k + M + P) + (P + Q + B_k),$$
$$(A_k + M + P) \cdot (P + Q + B_k) = P.$$

Since $K \supseteq L$,

$$K = K(A_k+M+P)+K(P+Q+B_k)$$

is a decomposition of K into two closed proper sub-sets whose common part is a continuum. This is a contradiction, by reference (H). There are, then, two sub-cases to discuss: (*a*) when L is the union of two indecomposable continua; (*b*) when L is indecomposable.

(*a*) Let $L=M+N$, where M and N are indecomposable and $M \cdot B_k = N \cdot A_k = 0$. Reasoning similar to the above shows that in this case K contains no points not on L. Let x be a point of $M \cdot N$; then it is not a point of A_k or B_k. It is easily seen that A_k+M is irreducible between a and x, and $N+B_k$ between x and b. Then, since M and N are indecomposable, M and N are the oscillatory sets of A_k+M and $N+B_k$ respectively about x. Hence

$$K = L = M+N$$

is the oscillatory set of C about x.†

(*b*) Let L be indecomposable. From the definition of oscillatory sets it is evident that L is the oscillatory set of C about any point of $C-(A_k+B_k)$; hence the theorem holds if $K=L$. If $K \neq L$, it follows by reasoning similar to

* See reference (D), p. 156.

† See reference (D), p. 153.

that used above that either $(K-L)\cdot A_k$ or $(K-L)\cdot B_k$ is void, say the latter. Then $K \subseteq A_k+L$. Let y be a point of $A_k \cdot L$; then A_k is irreducible between a and y, and $L+B_k$ is irreducible between y and b. Let Y' be the oscillatory set of A_k about y. Since y does not lie on B_k, the oscillatory set of $L+B_k$ about y is L. Then $Y=Y'+L$ is the oscillatory set of C about y.*

If $K \subseteq Y$, the theorem is proved; if not, we arrive at a contradiction. For K will contain a point z on A_k-Y'. Then, if Y_a' is the saturated semi-continuum of A_k-Y' containing a, $\overline{Y}_a' \supset A_k-Y' \supset z$.† Then $\overline{Y}_a'=A_k$, since A_k is irreducible between a and K. Hence Y' is complete‡ and z is a point of Y_a'. As K joins Y_a' and B_k, this shows that $K \supset Y'$. Then $K \cdot A_k$ and $Y'+L$ are closed proper parts of K and their divisor is the continuum Y'. This is a contradiction, as their union is K, which is an irreducible $S(m, n)$.

Thus the proof is complete, except for the special cases where K contains a or b, or both. The above demonstration holds for these cases, if we merely replace A_k by a if K contains a and B_k by b if K contains b.

8. THEOREM III. *Let F be the union of two bounded continua C_1 and C_2 having these properties: $C_1 \cdot C_2=\alpha+\beta$, where α and β are closed and $\alpha \cdot \beta=0$; both C_1 and C_2 are irreducible between α and β; and either C_1 and C_2 are both decomposable, or C_1 is indecomposable and C_2 is decomposable and is not the union of two indecomposable continua. Then the frontier of each secondary region determined by F is a part or the whole of some oscillatory set of C_1 or C_2, or it is a part or the whole of the union of the oscillatory sets of C_1 and C_2 about some point of α or β.*

PROOF. Let R' be any secondary region and let F' be its frontier. Let A_1, A_2, B_1, and B_2 be the oscillatory sets of C_1

* See referrence (D), p. 153.
† See reference (C), §10.
‡ See reference (C), §§15.

and C_2 about a point a of α and a point b of β, respectively.

It was shown in §6 that, if F' is not a part of A_1+A_2 or B_1+B_2, then F' is a part of C_1 or C_2. Since C_1 and C_2 are each irreducible between α and β, each of them is irreducible between a point a of α and a point b of β. Hence, if $F' \subseteq C_1$, the theorem of §7 shows that F' is a part of some oscillatory set of C_1. Thus the theorem is proved.

9. *Conclusion.* A consequence of the previous theorem is that under the hypotheses there stated the frontier of a secondary region is a part of either a continuum of condensation, an indecomposable continuum, a pair of indecomposable continua, or the union of a continuum of condensation and an indecomposable continuum. This follows at once from the proof of §7 and the fact that the oscillatory set of a bounded irreducible continuum about one of its end points is either a continuum of condensation or an indecomposable continuum.* Obviously there are no secondary regions unless A_1+A_2, or B_1+B_2, or some oscillatory set of C_1 or C_2 cuts the plane.

The conditions imposed on the character of the continua C_1 and C_2 in §5 take care of all possible cases with two exceptions. These are that both continua are indecomposable or that one is indecomposable and the other is the union of two indecomposable continua. With regard to the first it has been shown by Kuratowski† and Knaster‡ that the extended theorem of Rosenthal (§5) need not hold in this case. Whether or not it holds in the second case remains to be proved.

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* See reference (D), p. 155.

† C. Kuratowski, *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6, p. 138.

‡ B. Knaster, *Quelques coupures singulières du plan*, Fundamenta Mathematicae, vol. 7, p. 281.

ON THE MAPPING OF THE SEXTUPLES OF THE SYMMETRIC SUBSTITUTION GROUP G_6 IN A PLANE UPON A QUADRIC*

BY ARNOLD EMCH

1. *Introduction.* The six permutations of three elements x_1, x_2, x_3 considered as projective coordinates in a plane determine an involution of sextuples of points which may be mapped on a rational surface.† I shall show that in case of the involution thus defined the map is a quadric whose relation with the plane, established with sufficient details, will lead to some interesting geometric applications. The map of every configuration on the quadric will be a configuration in the plane, invariant under the G_6, whose geometric properties have been investigated before.‡

2. *Mapping of the G_6.* Let ϕ_1, ϕ_2, ϕ_3 represent the elementary symmetric functions $\phi_1=x_1+x_2+x_3$, $\phi_2=x_2x_3+x_3x_1+x_1x_2$, $\phi_3=x_1x_2x_3$, so that the general symmetric function of degree four has the form

$$y_i = a_i\phi_1^4 + b_i\phi_1^2\phi_2 + c_i\phi_1\phi_3 + d_i\phi_2^2, \tag{1}$$

depending upon three effective constants. Hence, there are four linearly independent functions y_i; $i=1, 2, 3, 4$, which we may set proportional to the four projective coordinates of a point in a space S_3. Thus to every point in (x), and consequently to every sextuple $I_6\{(x_1x_2x_3), (x_1x_3x_2), (x_2x_1x_3), (x_2x_3x_1), (x_3x_1x_2), (x_3x_2x_1)\}$, corresponds in S_3 a point (y), and as the system of sextuples is a continuous ∞^2 manifold, the locus of such points (y) must be a surface, which will be proved to be a quadric cone Q. For the y_i's we may

* Presented to the Society, September 9, 1927.

† Castelnuovo, G., *Sulla razionalità delle involuzioni piane*, Mathematische Annalen, vol. 44 (1894), pp. 125–155.

‡ Emch, A., *Some geometric applications of symmetric substitution groups*, American Journal, vol. 45 (1923), pp. 192–207.

evidently choose any four linearly independent symmetric quartics. For every choice of four such functions we obtain a certain surface Q. But all these surfaces are obviously collinearly related. The simplest choice is

$$(2)\quad \begin{cases} \rho y_1 = x_1^4 + x_2^4 + x_3^4 = \phi_1^4 - 4\phi_1^2\phi_2 + 4\phi_1\phi_3 + 2\phi_2^2, \\ \rho y_2 = x_2^2 x_3^2 + x_3^2 x_1^2 + x_1^2 x_2^2 = \phi_2^2 - 2\phi_1\phi_3, \\ \rho y_3 = x_1^3(x_2 + x_3) + x_2^3(x_3 + x_1) + x_3^3(x_1 + x_2) \\ \qquad = \phi_1^2\phi_2 - 2\phi_2^2 - \phi_1\phi_3, \\ \rho y_4 = x_1^2 x_2 x_3 + x_2^2 x_3 x_1 + x_3^2 x_1 x_2 = \phi_1\phi_3. \end{cases}$$

A simple elimination process of ϕ_1, ϕ_2, ϕ_3 leads to the required relations between the y's:

$$(3)\quad y_1(y_2 + 2y_4) + 2y_2^2 - y_3^2 - y_4^2 + 4y_2y_4 - 2y_3y_4 = 0,$$

which is a quadric cone Q with the vertex $V(4, -2, -1, -1)$, as can easily be verified.

To a plane section of Q corresponds in (x) a quartic which may be any of the reducible or irreducible types (1). Of particular importance are, of course, the exceptional elements of the (1, 6) correspondence between Q and (x). In the first place for the intersections $I(1, \omega, \omega^2)$, $J(1, \omega^2, \omega)$ of $\phi_1=0$ and $\phi_2=0$ there is $y_1=y_2=y_3=y_4=0$, so that I and J are fundamental points in (x). To the first neighborhoods of I and J, $(1+\alpha_1, \omega+\alpha_2, \omega^2+\alpha_3)$ and $(1+\alpha_1, \omega^2+\alpha_2, \omega+\alpha_3)$, correspond on Q for $\alpha_1+\alpha_2+\alpha_3 \neq 0$, $\rho y_1=4(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_2=-2(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_3=-(\alpha_1+\alpha_2+\alpha_3)$, $\rho y_4=\alpha_1+\alpha_2+\alpha_3$, or the point $V(4, -2, -1, 1)$. To a sextuple on $\phi_1=0$, distinct from I and J, corresponds on Q the point $T(2, 1, -2, 0)$. A plane $p=y_1+\lambda y_2+\mu y_3+\nu y_4=0$ cuts Q in a conic K and the join VT in a point R on K, to which corresponds in (x) the quartic

$$(4)\quad \phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (4 - 2\lambda - \mu + \nu)\phi_1\phi_3 + (2 + \lambda - 2\mu)\phi_2^2 = 0,$$

which has $\phi_1=0$ as a double tangent. To a generic point R on VT correspond thus the first neighborhoods of I and J on $\phi_1=0$. To a plane

(5) $$y_1 + \lambda y_2 + \mu y_3 + (2\lambda + \mu - 4)y_4 = 0$$

through V, corresponds the quartic

(6) $$\phi_1^4 + (\mu - 4)\phi_1^2\phi_2 + (2 + \lambda - 2\mu)\,\phi_2^2 = 0,$$

which clearly reduces to the product of two conics of the type $\phi_1^2 + k\phi_2 = 0$. Thus a generic plane of the bundle through V cuts Q in two generatrices to which correspond in (x) two conics of the symmetric pencil $\phi_1^2 + k\phi_2 = 0$. To a tangent plane of Q corresponds a double conic $(\phi_1^2 + k\phi_2)^2 = 0$. The tangent plane at R

$$y_1 + 6y_2 + 4y_3 + 12y_4 = 0$$

touches Q along VT, and to this intersection of the tangent plane, VT counted twice, corresponds in (x) the quadruple line $\phi_1^4 = 0$.

3. *Mapping of Intersections of the Quadric Cone.* A generic surface F_n cuts Q in a space curve C_{2n} to which corresponds in (x) a symmetric curve C'_{4n} (curve in which the coordinates enter symmetrically). As F_n cuts VT in n points, $\phi_1 = 0$ is, in general, an M-fold double tangent of C'_{4n}. This also appears directly from the fact that in (1) ϕ_2^2 appears in y_1, y_2, y_3, and ϕ_1 is a factor of all other terms in which ϕ_2^2 is not contained.

Conversely to a symmetric n-ic C'_n in (x) corresponds on Q a curve, whose order can easily be determined in every case. For instance, when C'_n does not pass through I and J, which is the case when C'_n contains the term ϕ_3^m, $3m = n$, then a generic quartic C'_4 cuts C'_n in $12m$ points which form $2m$ sextuples. To these correspond on Q, $2m$ points which lie on a plane of the corresponding conic K of C_4, and which are the intersections of the curve C on Q, corresponding to C'_4. The order of C is therefore $2m$. The curve C_{2m} on Q is cut out by a surface F_m, which may possibly pass through generatrices of Q, or through the point T. For example when C'_n is a sextic, then F_m is a quadric which passes through a generatrix of Q, or is a quadric cone with its

vertex at T. This is in agreement with the counting of constants. The number $N+1$ of terms of a symmetric n-ic is equal to the number of positive integral solutions of the diophantine equation $\alpha+2\beta+3\gamma=n$ resulting from the general term $\phi_1^{\alpha}\phi_2^{\beta}\phi_3^{\gamma}$ of the n-ic, and is equal to the nearest positive integer contained in $(n+3)^2/12$. For $n=6$, $N+1=7$. The number of constants in F_2 is 10 which is reduced to 7 by the condition to pass through a generatrix of Q. To the intersection of F_2 with Q corresponds in (x) an octavic. But to the generatrix of Q, common to F_2, corresponds a symmetric conic as a factor of the octavic, so that a sextic remains as a residual curve.

More generally F_m cuts Q in a curve to which corresponds in (x) a curve of order $4m$. In order that this reduce to $3m$ it is necessary that a factor of order m split off. These factors are of the form $\phi_1^{\alpha}\phi_2^{\beta}$, with $\alpha+\beta=m$, and F_m must pass α times through R and contains β generatrices of Q. Thus in case of $n=9$, $m=3$, F_m must be either a cubic cone with vertex at T, or a cubic surface through T and a generatrix of Q. The condition to pass through T and a generatrix of Q absorbs 5 constants of F_3, and leaves 15 (14 effective) disposable constants. But through the intersection of a quadric and a cubic surface there are ∞^3 other cubics, so that there are ∞^{12} linearly independent residual quintics on Q to which corresponds in (x) the same manifold of conics, which is in agreement with the number of solutions of the diophantine equation, in case of $n=9$.

4. *Symmetric Quartics.* To a pencil of planes through a line s cutting Q in A and B corresponds in (x) a pencil of quartics with the same double tangent $\phi_1=0$ and with two sextuples A' and B', corresponding to A and B, as base-points outside of the double tangencies at I and J. When s is tangent to Q, the quartics of the pencil all touch each other in the points of a sextuple. Now consider any two conics K' and K'' on Q. The common tangent-planes of K' and K'' envelope two cones. Through a generic

point of Q there are two tangent-planes to each of these cones. To K' and K'' correspond in (x) two quartics C_4' and C_4''. Every tangent plane of one of these cones cuts Q in a conic which touches K' and K''. To this conic corresponds a quartic which touches each C_4' and C_4'' in points of a sextuple, one for each C_4' and C_4''. Hence we may state the following theorem.

THEOREM. *Given two symmetric quartics C_4' and C_4'', then there exist two ∞^1 systems of symmetric quartics of index 2, such that every quartic of each system has a sextuple contact (contact in points of a sextuple) with each C_4' and C_4''.*

To the intersection of a plane through T with Q corresponds in (x) a symmetric cubic $\phi_1^3+\lambda\phi_1\phi_2+\mu\phi_3=0$. Let K' again be a conic not through I. Now I is the vertex of a quadric cone through K', whose tangent planes cut Q in conics tangent to K'. To these correspond in (x) cubics and a quartic respectively. This leads to the following theorem.

THEOREM. *For a symmetric quartic corresponding to a generic conic on Q there exists a system of cubics of index 2 with the property of sextuple contacts with the quartic.**

5. *A Problem in Closure.* Let K' and K'' be again two conics on Q not intersecting in real points and O a generic point in space. O as a vertex determines with K' and K'' two cones C' and C'', which intersect Q in two other conics L' and L''. Assume O such that C' and C'' have no real generatrix in common, moreover so that it is possible to construct a closed pyramid of n faces inscribed to one cone, say C', and circumscribed to C''. To the four conics K', K'', L', L'' correspond in (x) four quartics C_4', C_4'', D_4', D_4''. To a conic cut out on Q by a face of P corresponds in (x) a quartic which cuts each C_4' and D_4' in a sextuple and touches each C_4'' and D_4'' in points of a sextuple. As there

* Such systems of sextuple tact cubics for the general quartic were established by A. Clebsch, Mathematische Annalen, vol. 3 (1871), pp. 45–75.

are ∞^1 such inscribed and circumscribed pyramids P the mapping upon (x) gives the following theorem.

THEOREM. *Let C_4' and C_4'' be two fixed symmetric quartics in (x). Construct a quartic $C_4^{(1)}$ with a sextuple contact with C_4'', cutting C_4' in two sextuples S_1 and S_2. Through S_2 draw another quartic $C_4^{(2)}$ with a sextuple contact with C_4'', which cuts C_4' in another sextuple S_3. Through S_3 draw similarly a third quartic $C_4^{(3)}$, cutting C_4' in a sextuple S_4, and so forth. Suppose that after drawing n such quartics, the last $C_4^{(n)}$ through S_n cuts C_4' in a sextuple S_{n+1} which coincides with S_1. If this happens once then there exists an infinite number of such series of quartics with the closure property.*

Moreover there exist two other fixed quartics D_4' and D_4'' which are related to these series in precisely the same manner as C_4' and C_4''.

THE UNIVERSITY OF ILLINOIS

CONGRUENCES OF LINES OF SPECIAL ORIENTATION RELATIVE TO A SURFACE OF REFERENCE*

BY M. C. FOSTER

1. *Introduction.* With each line l of a rectilinear congruence let us associate the point M in which l intersects a surface of reference S. We refer S to any orthogonal system. Let α, β, γ be the direction-cosines of l relative to the moving trihedral of S at M, the x-axis being chosen tangent to the curve $v=\text{const.}$ By congruences of special orientation relative to S, we shall mean those congruences for which the functions α, β, γ are of a special form. The present paper is concerned primarily with the case when α, β, γ are constant.

2. *Normal Congruences.* Relative to the moving trihedral the coordinates of any point P on l are

* Presented to the Society, December 28, 1926.

$$x = \alpha t, \quad y = \beta t, \quad z = \gamma t, \tag{1}$$

where t is the distance along l from M to P. The condition that there exist a surface Σ normal to l is that the displacements of some point on l satisfy the relation

$$\sum \alpha \delta x = 0, \tag{2}$$

for all values of dv/du. This becomes on using the well known formulas*

$$\frac{\partial t}{\partial u} du + \frac{\partial t}{\partial v} dv + \alpha\xi \, du + \beta\eta_1 \, dv \equiv 0. \tag{3}$$

Hence

$$\frac{\partial t}{\partial u} + \alpha\xi = 0, \qquad \frac{\partial t}{\partial v} + \beta\eta_1 = 0. \tag{4}$$

The condition of integrability,

$$\frac{\partial}{\partial v}(\alpha\xi) = \frac{\partial}{\partial u}(\beta\eta_1), \tag{5}$$

is therefore a necessary condition that the congruence be normal. It is also sufficient. For if (5) is satisfied, the function t as given by (4) will satisfy (3), and consequently there exists a one-parameter family of surfaces Σ normal to the congruence.

Let us now assume that

$$\alpha = U, \quad \beta = V, \tag{6}$$

where U and V are functions of u and v alone respectively. The relation (5) becomes

$$U\frac{\partial \xi}{\partial v} = V\frac{\partial \eta_1}{\partial u},$$

which may be written†

$$U\eta_1 r = -V\xi r_1, \tag{7}$$

* Eisenhart, *Differential Geometry of Curves and Surfaces*, p. 170.

† Eisenhart, p. 170.

or

$$-\frac{U}{V}=\frac{\xi r_1}{\eta_1 r}=\frac{\rho_{gu}}{\rho_{gv}}, \tag{8}$$

where ρ_{gu} and ρ_{gv} are the radii of geodesic curvature of the curves $v=$const. and $u=$const., respectively. From (8) we have the following theorem.

THEOREM 1. *If the lines l be of fixed orientation relative to the trihedral of S, a necessary and sufficient condition that the congruence be normal is that the corresponding radii of geodesic curvature of the curves $v=$const. and $u=$const., respectively, be in the constant ratio $\alpha:-\beta$.*

The condition that the curves $v=$const., and $u=$const., be geodesics is that $r=0$, and $r_1=0$, respectively. Hence from (7) we have the following theorems.*

THEOREM 2. *If α, (β), be zero, a necessary and sufficient condition that the congruence be normal is that the curves $u=$const., $(v=$const.$)$, be geodesics.*

THEOREM 3. *If the curves $u=$const., $(v=$const.$)$, be geodesics, a necessary and sufficient condition that the congruence be normal is that either the curves $v=$const., $(u=$const.$)$, be geodesics and S be developable, or α, (β), be zero.*

We note that Theorem 2 includes as a special case the well known theorem that a necessary and sufficient condition that the tangents to a family of curves on a surface constitute a normal congruence is that these curves be geodesics.

3. *Equation Defining the Developables.* We shall assume for the moment that α, β, γ are any functions whatever of u and v. For l to generate a developable surface the displacement of some point on l must satisfy the relations

$$\frac{\delta x}{\alpha}=\frac{\delta y}{\beta}=\frac{\delta z}{\gamma},$$

or

$$\alpha\delta z-\gamma\delta x=0, \qquad \beta\delta z-\gamma\delta y=0. \tag{9}$$

* We exclude the case where the lines are normal to S.

When t is eliminated between these equations we get the following equation for the curves on S defining the developables,

$$(10)\qquad \begin{vmatrix} \alpha & \xi du & A_1du + A_2dv \\ \beta & \eta_1 dv & B_1du + B_2dv \\ \gamma & 0 & C_1du + C_2dv \end{vmatrix} = 0,$$

where we have set

$$(11)\qquad \begin{cases} A_1 = \dfrac{\partial \alpha}{\partial u} + q\gamma - r\beta, & A_2 = \dfrac{\partial \alpha}{\partial v} + q_1\gamma - r_1\beta, \\ B_1 = \dfrac{\partial \beta}{\partial u} + r\alpha - p\gamma, & B_2 = \dfrac{\partial \beta}{\partial v} + r_1\alpha - p_1\gamma, \\ C_1 = \dfrac{\partial \gamma}{\partial u} + p\beta - q\alpha, & C_2 = \dfrac{\partial \gamma}{\partial v} + p_1\beta - q_1\alpha. \end{cases}$$

When dv/du is eliminated between equations (9) we get the following equation in t for the distances to the focal points:

$$(12)\qquad \begin{vmatrix} \alpha & A_1t + \xi & A_2t \\ \beta & B_1t & B_2t + \eta_1 \\ \gamma & C_1t & C_2t \end{vmatrix} = 0.$$

4. *Distance to a Line of Striction.* The direction-cosines

$$\alpha + \delta\alpha, \quad \beta + \delta\beta, \quad \gamma + \delta\gamma$$

of a neighboring line l' of the congruence are found by considering the displacements $\delta\alpha$, $\delta\beta$, $\delta\gamma$ of a point (α, β, γ) on the unit sphere, relative to a trihedral of fixed vertex at the center of the unit sphere, whose axes are parallel to the axes of the moving trihedral. If a point P on l generate the line of striction of the ruled surface defined by a value of dv/du, the displacement of P must be orthogonal to both l and l'. Hence

$$\sum \alpha\delta x = 0, \qquad \sum (\alpha + \delta\alpha)\delta x = 0,$$

or

$$\sum \delta\alpha\delta x = 0.$$

This becomes, by means of well known formulas,

$$
\begin{aligned}
(13)\quad & \left\{\frac{\partial\alpha}{\partial u}du + \frac{\partial\alpha}{\partial v}dv + \gamma(q\,du + q_1 dv) - \beta(r\,du + r_1\,dv)\right\} \\
& \cdot \left\{\left(\frac{\partial\alpha}{\partial u}du + \frac{\partial\alpha}{\partial v}dv\right)t + \xi\,du + \gamma t(q\,du + q_1 dv)\right. \\
& \left. - \beta t(r\,du + r_1\,dv)\right\} \\
& + \left\{\frac{\partial\beta}{\partial u}du + \frac{\partial\beta}{\partial v}dv + \alpha(r\,du + r_1\,dv) - \gamma(p\,du + p_1\,dv)\right\} \\
& \cdot \left\{\left(\frac{\partial\beta}{\partial u}du + \frac{\partial\beta}{\partial v}dv\right)t + \eta_1 dv + \alpha t(r\,du + r_1\,dv)\right. \\
& \left. - \gamma t(p\,du + p_1\,dv)\right\} \\
& + \left\{\frac{\partial\gamma}{\partial u}du + \frac{\partial\gamma}{\partial v}dv + \beta(p\,du + p_1 dv) - \alpha(q du + q_1 dv)\right\} \\
& \cdot \left\{\left(\frac{\partial\gamma}{\partial u}du + \frac{\partial\gamma}{\partial v}\right)t + \beta t(p\,du + p_1\,dv)\right. \\
& \left. - \alpha t(q\,du + q_1\,dv)\right\} = 0.
\end{aligned}
$$

If a value be assigned to t this equation gives us the equation of the curves on S defining the ruled surfaces for which t is the distance to their lines of striction; and if a value be assigned to dv/du we have an equation in t which gives us the distance to the line of striction of the ruled surface so determined.

5. *The Direction Cosines α, β, γ, constant.* We now assume the lines of the congruence to be of fixed orientation relative to the moving trihedral. The curves defining the ruled surfaces whose lines of striction lie on S will be found by setting $t = 0$ in (13). We get, for the equation of these curves,

$$\xi(\beta r - \gamma q)du^2 + \{\xi(\beta r_1 - \gamma q_1) + \eta_1(\gamma p - \alpha r)\}du\,dv + \eta_1(\gamma p_1 - \alpha r_1)dv^2 = 0. \tag{14}$$

The curves $v=$const., or $u=$const., will be the lines of striction of the ruled surfaces which they define if

$$\beta r - \gamma q = 0,$$

or

$$\gamma p_1 - \alpha r_1 = 0,$$

respectively. If

$$\beta r - \gamma q = 0,$$

we have*

$$\frac{\beta}{\gamma} = \text{const.} = \frac{q}{r} = -\operatorname{ctn}\omega_u, \tag{15}$$

where ω_u is the angle between the positive principal normal of the curve $v=$const., and the positive normal to S, the angle being measured toward the positive binormal. Hence we have the following theorem.

THEOREM 4. *If l be of fixed orientation relative to the trihedral, a necessary and sufficient condition that the curves $v=$const. be the lines of striction of the ruled surfaces which they define is that their osculating planes meet the normal under the constant angle defined by* (15).

Let us here recall a theorem of ruled surfaces due to Bonnet:

If a curve upon a ruled surface have two of the following properties, it has the third also:

1. that it cut the rulings under constant angle,
2. that it be the line of striction,
3. that it be a geodesic.

Hence Theorem 4 may be extended to state that the curves $v=$const. are geodesics on the ruled surfaces which they

* Eisenhart, p. 167.

define; and the principal normals meet the rulings orthogonally. A similar theorem holds for the curves $u=\text{const.}$

The condition that S be the middle surface is that the coefficient of t in (12) shall vanish. We get as this condition

$$\xi p_1(\alpha^2-1)+\eta_1 q(1-\beta^2)+\alpha\beta(\xi q_1-\eta_1 p)+\gamma(\alpha\xi r_1-\beta\eta_1 r)=0. \tag{16}$$

If S be minimal we have $\xi p_1-\eta_1 q=0$; consequently, if $\alpha=\beta$, and S be referred to its lines of curvature, (16) reduces to

$$\xi r_1=\eta_1 r,$$

or

$$\rho_{gu}=\rho_{gv}.$$

Hence we have the following theorem.

THEOREM 5. *If S be a minimal surface referred to its lines of curvature, a necessary and sufficient condition that the congruence of fixed orientation for which $\alpha=\beta$, have S as its middle surface, is that the corresponding radii of geodesic curvature of the parametric curves be equal.*

We note from (7) that such congruences are not normal.

Let the lines of curvature be parametric. The condition that the developables of the congruence be represented on S by the parametric system is that the coefficients of du^2 and dv^2 in (9) shall vanish. This condition is

$$\alpha C_2-\gamma A_2=0,\quad \gamma B_1-\beta C_1=0. \tag{17}$$

Hence

$$\begin{cases} -\dfrac{\alpha}{\gamma}=\text{const.}=\dfrac{r_1}{p_1}=\tan\omega_v, \\ \dfrac{\beta}{\gamma}=\text{const.}=-\dfrac{r}{q}=\tan\omega_u. \end{cases} \tag{18}$$

Consequently, the osculating planes of the lines of curvature meet the tangent planes under constant angle, and they are

therefore plane curves.* We have in consequence the following theorem.

THEOREM 6. *If S be referred to its lines of curvature, a necessary and sufficient condition that the developables of the congruence of lines of fixed orientation be defined by the parametric curves is that these curves be plane in both systems, and that their osculating planes meet the normals under the constant angles defined by* (18).

When relations (18) are substituted in (16), this condition that S be the middle surface reduces to

$$\xi p_1 - \eta_1 q = 0.$$

Hence if we further assume S to be the middle surface, it must also be minimal. We note that in this case the congruence is of the Ribaucour type; for the developables intersect the middle surface in a conjugate system.

6. *Isotropic Congruences.* We shall again assume for the moment that α, β, γ are any functions whatever of u and v. The necessary and sufficient condition that a congruence be isotropic is that all the lines of striction lie on the middle surface. Evidently if all lines of striction lie on some surface, that surface is the middle surface. We suppose that all lines of striction lie on S. Then from (13) we must have

$$t \equiv 0,$$

or

$$A_1 = 0, \quad \xi A_2 + \eta_1 B_1 = 0, \quad B_2 = 0. \tag{19}$$

Hence when S is the middle surface (19) represents the necessary and sufficient condition that a congruence be isotropic.

We again assume α, β, γ constant. The relations (19) become

$$\begin{cases} q\gamma - r\beta = 0, \\ \xi(q_1\gamma - r_1\beta) + \eta_1(r\alpha - p\gamma) = 0, \\ r_1\alpha - p_1\gamma = 0, \end{cases} \tag{20}$$

* Eisenhart, p. 150.

from which we have

$$(21)\qquad \begin{cases} \dfrac{\gamma}{\beta} = \text{const.} = \dfrac{r}{q} = -\tan\omega_u, \\ \dfrac{\gamma}{\alpha} = \text{const.} = \dfrac{r_1}{p_1} = \tan\omega_v. \end{cases}$$

Using the first and third equations of (20) the second member reduces to

$$(22)\qquad \eta_1 p_1 r^2 - \xi q r_1^2 + 2\xi q_1 r r_1 = 0.$$

From (21) we see that the osculating planes of the parametric curves meet the normals under constant angles. We have therefore the following theorem.

THEOREM 7. *A condition that the congruence of lines of fixed orientation relative to its middle surface be an isotropic congruence, is that the osculating planes of the parametric curves meet the normals under the constant angles defined by* (21), *and that the fundamental quantities of the middle surface satisfy* (22).

7. *Congruences of Ribaucour.* The condition that a congruence be of the Ribaucour type is that its developables meet the middle surface in a conjugate system. We assume S to be the middle surface referred to its lines of curvature. From (10) we readily find the condition that the curves defining the developables form a conjugate system; this reduces to

$$(23)\qquad p_1 r\alpha - q r_1 \beta = 0.$$

Hence when S is taken as the middle surface referred to its lines of curvature, the relation (23) represents the necessary and sufficient condition that a congruence of fixed orientation relative to S be of the Ribaucour type. This condition is readily reducible to

$$\frac{\alpha}{\beta} = \text{const.} = -\tan\bar{\omega}_v \operatorname{ctn}\bar{\omega}_u.$$

8. *Condition that the Lines of Curvature on S and on the Normal Surfaces be in Correspondence.* We have seen in §5 that when the lines of curvature on S are parametric, a condition that the developables be represented by the parametric curves is

$$\frac{\alpha}{\gamma} = -\frac{r_1}{p_1}, \qquad \frac{\gamma}{\beta} = -\frac{q}{r}.$$

From these we get

$$\frac{\alpha}{\beta} = \frac{r_1 q}{p_1 r}. \tag{24}$$

When the congruence is normal we have, from (7),

$$\frac{\alpha}{\beta} = -\frac{\xi r_1}{\eta_1 r}; \tag{25}$$

and from (24) and (25) we have

$$\xi p_1 + \eta_1 q = 0. \tag{26}$$

Hence, when S is referred to its lines of curvature, the relation (26) represents a condition that the parametric curves on S correspond to the lines of curvature on the surfaces normal to the congruence of fixed orientation.

WILLIAMS COLLEGE

A NEW TABLE OF THE ZEROS OF THE BESSEL FUNCTIONS $J_0(x)$ AND $J_1(x)$ WITH CORRESPONDING VALUES OF $J_1(x)$ AND $J_0(x)$*

BY H. T. DAVIS AND W. J. KIRKHAM

1. *Introduction.* In connection with a problem involving the calculation of the distribution of energy in a diffraction pattern of sixty-six rings, it was found necessary to extend existing tables of the zeros of the Bessel functions $J_0(x)$ and $J_1(x)$ with corresponding values of $J_1(x)$ and $J_0(x)$. In the hope that these computations may be of use to other investigators they are reproduced here.

The first ten zeros in Table I and the first fifty zeros with the associated values of $J_0(x)$ in Table II are incorporated from tables by E. Meissel published in 1888 and 1889–90 respectively.† The values of the zeros of $J_0(x)$ and their logarithms from $s=11$ to $s=40$ have been taken from a paper by R. W. Willson and B. O. Peirce.‡ The associated values of $J_1(x)$ have been recomputed to ten decimal places and one correction in the Willson-Peirce table is noted, the value corresponding to $s=35$ being in error.

2. *Formulas Used in the Calculation of the Zeros of $J_0(x)$ and $J_1(x)$.* In computing the values of $x_0^{(s)}$ and $x_1^{(s)}$, the roots of $J_0(x)$ and $J_1(x)$ respectively, the formulas of G. C. Stokes were used.§ Thus we have

$$x_0^{(s)} = \frac{1}{4}\pi a\left(1+\frac{\alpha_2}{a^2}+\frac{\alpha_4}{a^4}+\frac{\alpha_6}{a^6}+\frac{\alpha_8}{a^8}\right), \qquad a = 4s-1,$$

* These tables were made possible by a grant of funds from the Waterman Institute, Indiana University.

† Mathematische Abhandlungen der Akademie der Wissenschaften zu Berlin, 1888; Annual report of the Ober-Realschule at Kiel, 1889–90.

‡ This Bulletin, vol. 3 (1897), pp. 153–155.

§ Cambridge Philosophical Transactions, vol. 9, or Mathematical and Physics Papers, vol. II, pp. 353, 355. See also J. McMahon, Annals of Mathematics, vol. 9 (1894–95), p. 25.

where

$$\alpha_2 = 2/\pi^2, \quad \alpha_4 = -62/(3\pi^4), \quad \alpha_6 = 15116/(15\pi^6),$$

$$\alpha_8 = -12554474/(105\pi^8);$$

$$x_0^{(s)} = \pi s - .7853981634 + \frac{k_1}{a} - \frac{k_2}{a^3} + \frac{k_3}{a^5} - \frac{k_4}{a^7},$$

where

$$\log k_1 = 9.2018201316, \qquad \log k_2 = 9.2217608253,$$

$$\log k_3 = 9.9155362692, \qquad \log k_4 = 0.9955001222.$$

$$x_1^{(s)} = \frac{1}{4}\pi b\left(1 + \frac{\beta_2}{b^2} + \frac{\beta_4}{b^4} + \frac{\beta_6}{b^6} + \frac{\beta_8}{b^8}\right), \qquad b = 4s + 1,$$

where

$$\beta_2 = -6/\pi^2, \quad \beta_4 = 6/\pi^4, \quad \beta_6 = -4716/(5\pi^6),$$

$$\beta_8 = 3902418/(35\pi^8);$$

$$x_1^{(s)} = \pi s + .7853981634 - \frac{m_1}{b} + \frac{m_2}{b^3} - \frac{m_3}{b^5} + \frac{m_4}{b^7},$$

where

$$\log m_1 = 9.6789413863, \qquad \log m_2 = 8.6846416409,$$

$$\log m_3 = 9.8867944373, \qquad \log m_4 = 0.9651566416.$$

The values of $x_0^{(s)}$ and $x_1^{(s)}$ were calculated from these formulas by means of Vega's ten-place table of logarithms. In checking the zeros, the differences between successive values were computed on a ten-bank Monroe calculator with the aid of Barlow's table of reciprocals and cubes from the following formulas:

$$x_0^{(s+1)} - x_0^{(s)} = \pi - .1591549431[1/(4s-1) - 1/(4s+3)] + .1666329280[1/(4s-1)^3 - 1/(4s+3)^3],$$

$$x_1^{(s+1)} - x_1^{(s)} = \pi + .4774648293[1/(4s+1) - 1/(4s+5)] - .0483773016[1/(4s+1)^3 - 1/(4s+5)^3].$$

3. *The Calculation of* $J_1[x_0^{(s)}]$ *and* $J_0[x_1^{(s)}]$. For the calculation of $J_1[x_0^{(s)}]$ and $J_0[x_1^{(s)}]$ new asymptotic formulas were used. It is well known* that $J_0(x)$ and $J_1(x)$ have the following asymptotic expansions:

$$(1)\quad \begin{cases} J_0(x) = \left(\dfrac{2}{\pi x}\right)^{1/2}[P_0 \cos(x-\pi/4) + Q_0 \sin(x-\pi/4)], \\ J_1(x) = \left(\dfrac{2}{\pi x}\right)^{1/2}[P_1 \cos(x-\pi/4) + Q_1 \sin(x-\pi/4)], \end{cases}$$

where

$$P_0 = 1 - \frac{1^2 3^2}{2!(8x)^2} + \frac{1^2 3^2 5^2 7^2}{4!(8x)^4} - \cdots,$$

$$Q_0 = \frac{1^2}{1!8x} - \frac{1^2 3^2 5^2}{3!(8x)^3} + \frac{1^2 3^2 5^2 7^2 9^2}{5!(8x)^5} - \cdots,$$

$$P_1 = \frac{3}{1!8x} + \frac{3\cdot 5\cdot 21}{3!(8x)^3} + \frac{3\cdot 5\cdot 21\cdot 45\cdot 77}{5!(8x)^5} - \cdots,$$

$$Q_1 = 1 + \frac{3\cdot 5}{2!(8x)^2} - \frac{3\cdot 5\cdot 21\cdot 45}{4!(8x)^4} + \cdots.$$

If in the formulas (1) we let $x = x_1^{(s)}$ and use the identity $\sin^2(x-\pi/4) + \cos^2(x-\pi/4) = 1$ to eliminate the sine and cosine terms, we get the following asymptotic formula for the calculation of $J_0(x_1^{(s)})$:

$$(2)\quad \begin{aligned} J_0(x_1^{(s)}) &= \pm\left(\frac{2}{\pi x_1}\right)^{1/2}\left[1 + \frac{1\cdot 3}{2!(2x_1)^2} - \frac{1^4 3^3 5}{4!(2x_1)^4} \right. \\ &\qquad \left. + \frac{1^4 3^4 5^3 7}{6!(2x_1)^6} - \cdots\right]^{-1/2}, \\ &= \pm\left(\frac{2}{\pi x_1}\right)^{1/2}\left[1 - \frac{3}{2^4 x_1^2} + \frac{3^2 13}{2^9 x_1^4} - \frac{3^2\cdot 5\cdot 7\cdot 23}{2^{13} x_1^6} \right. \\ &\qquad \left. + \frac{3^4\cdot 5\cdot 10487}{2^{19} x_1^8} + \cdots\right]. \end{aligned}$$

* See Gray, Mathews and MacRobert, *A Treatise on Bessel Functions*, London, 1922, pp. 57–59.

The sign is plus when s is even, and is minus when s is odd.

Similarly, replacing x by $x_0^{(s)}$ and eliminating the sine and cosine terms from (1) we obtain the following formula for $J_1(x_0^{(s)})$:

$$
\begin{aligned}
J_1(x_0^{(s)}) &= \pm \left(\frac{2}{\pi x_0}\right)^{1/2}\left[1 - \frac{1^4}{2!(2x_0)^2} + \frac{1^4 3^4}{4!(2x_0)^4}\right. \\
&\qquad \left. - \frac{1^4 3^4 5^4}{6!(2x_0)^6} + \cdots\right]^{-1/2}, \\
&= \pm \left(\frac{2}{\pi x_0}\right)^{1/2}\left[1 + \frac{1}{2^4 x_0^2} - \frac{3 \cdot 17}{2^9 x_0^4} + \frac{43 \cdot 101}{2^{13} x_0^6}\right. \\
&\qquad \left. - \frac{3025837}{2^{19} x_0^8} + \cdots\right].
\end{aligned} \tag{3}
$$

The sign is plus when s is odd, and is minus when s is even.

If in (2) we substitute the value of $x_1^{(s)}$ from Stokes' formula, and in (3) the value of $x_0^{(s)}$, we obtain new formulas which are remarkable in that terms involving the reciprocals of $(4s+1)^2$ and $(4s-1)^2$ are absent:

$$
J_0(x_1^{(s)}) = \pm \frac{2 \cdot 2^{1/2}}{\pi b^{1/2}}\left[1 + \frac{24}{\pi^4 b^4} - \frac{19584}{10\pi^6 b^6} + \frac{2466720}{7\pi^8 b^8}\right], \qquad b = 4s + 1,
$$

$$
J_1(x_0^{(s)}) = \pm \frac{2 \cdot 2^{1/2}}{\pi a^{1/2}}\left[1 - \frac{56}{3\pi^4 a^4} + \frac{9664}{5\pi^6 a^6} - \frac{7381280}{21\pi^8 a^8}\right], \qquad a = 4s - 1.
$$

From these formulas we see that to ten decimal places

$$
J_0(x_1^{(s)}) = \frac{2 \cdot 2^{1/2}}{\pi} \frac{1}{(4s+1)^{1/2}} \text{ and } J_1(x_0^{(s)}) = \frac{2 \cdot 2^{1/2}}{\pi} \frac{1}{(4s-1)^{1/2}}
$$

for values of s greater than 10. In the calculation of $J_0(x_1^{(s)})$ and $J_1(x_0^{(s)})$ from these formulas Vega's ten-place table of logarithms was used. Each value was checked by direct computation on the Monroe calculator. The first five values of $J_1(x_0^{(s)})$ were reduced from twelve-place values obtained by interpolation from Meissel's table.

4. *Formulas for the Calculation of $J_n(x)$ from Tables of $J_0(x)$ and $J_1(x)$.* It should be noted that corresponding values of $J_n(x)$ can be obtained from tables of $J_0(x)$ and $J_1(x)$ by the well known recurrence formula:

$$\frac{2nJ_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x).$$

However, when n is large, the labor of calculation becomes very great. The following formulas, which, as far as the authors are aware, are not found in standard treatises on the Bessel functions, are very useful in the calculation of $J_n(x)$ from tables of $J_0(x)$ and $J_1(x)$:

$$J_{2n}(x) = (-1)^n\left[1 + \sum_{r=1}^{n-1}(-1)^r \frac{2^{2r}n^2(n^2-1)^2\cdots}{(2r)!x^{2r}} \cdot \frac{[n^2-(r-1)^2]^2(n^2-r^2)}{1}\right]J_0(x)$$
$$+ (-1)^{n+1}\left[\sum_{r=0}^{n-1}(-1)^r \frac{2^{2r+1}n^2(n^2-1)^2(n^2-4)^2}{(2r+1)!x^{2r+1}} \cdot \frac{\cdots(n^2-r^2)^2}{1}\right]J_1(x).$$

$$J_{2n-1}(x) = (-1)^{n+1}\left[\sum_{r=0}^{n-2}(-1)^r \frac{2^{2r+1}n^2(n^2-1)^2\cdots}{(2r+1)!x^{2r+1}} \cdot \frac{[n^2-(r-1)^2]^2(n^2-r^2)(n-r)(n-r-1)}{1}\right]J_0(x)$$
$$+ (-1)^{n+1}\left[1 + \sum_{r=1}^{n-1}(-1)^r \frac{2^{2r}n^2(n^2-1)^2\cdots}{(2r)!x^{2r}} \cdot \frac{[n^2-(r-1)^2]^2(n-r)^2}{1}\right]J_1(x).$$

These formulas are easily proved by induction.

TABLE I.

THE FIRST 150 ROOTS OF $J_0(x)$ WITH CORRESPONDING VALUES OF $J_1(x)$

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
1	2.4048255577	0.3810835788	+0.5191474973	9.715290765
2	5.5200781103	0.7419452231	−0.3402648065	9.531817033
3	8.6537279129	0.9372032361	+0.2714522999	9.433693526
4	11.7915344391	1.0715703238	−0.2324598214	9.366347900
5	14.9309177086	1.1740865018	+0.2065464331	9.315017699
6	18.0710639679	1.2569837232	−0.1877288030	9.273530911
7	21.2116366299	1.3265741787	+0.1732658943	9.238713084
8	24.3524715308	1.3865430443	−0.1617015504	9.208714184
9	27.4934791320	1.4392297006	+0.1521812139	9.182361044
10	30.6346064684	1.4862123057	−0.1441659779	9.158862782
11	33.7758202136	1.5286059043	+0.1372969435	9.137660869
12	36.9170983537	1.5672275586	−0.1313246267	9.118346175
13	40.0584257646	1.6026938781	+0.1260694971	9.100610020
14	43.1997917132	1.6354816528	−0.1213986249	9.084213767
15	46.3411883717	1.6659671666	+0.1172111989	9.068969108
16	49.4826098974	1.6944525978	−0.1134291926	9.054724841
17	52.6240518411	1.7211842839	+0.1099911432	9.041357716
18	55.7655107550	1.7463656842	−0.1068478884	9.028765943
19	58.9069839261	1.7701667872	+0.1039595729	9.016864487
20	62.0484691902	1.7927310714	−0.1012934991	9.005581573
21	65.1899648002	1.8141807465	+0.0988225538	8.994856073
22	68.3314693299	1.8346207594	−0.0965240405	8.984635493
23	71.4729816036	1.8541418997	+0.0943787940	8.974874423
24	74.6145006437	1.8728232368	−0.0923705049	8.965533317
25	77.7560256304	1.8907340543	+0.0904851942	8.956577523
26	80.8975558711	1.9079354006	−0.0887108024	8.947976507
27	84.0390907769	1.9244813451	+0.0870368634	8.939703232
28	87.1806298436	1.9404200023	−0.0854542429	8.931733631
29	90.3221726372	1.9557943757	+0.0839549293	8.924046200
30	93.4637187819	1.9706430570	−0.0825318613	8.916621640
31	96.6052679510	1.9850008094	+0.0811787883	8.909442565
32	99.7468198587	1.9988990584	−0.0798901543	8.902493260
33	102.8883742542	2.0123663047	+0.0786610017	8.895759473
34	106.0299309165	2.0254284784	−0.0774868911	8.889228237
35	109.1714896498	2.0381092361	+0.0763638333	8.882887721
36	112.3130502805	2.0504302219	−0.0752882326	8.876727102
37	115.4546126537	2.0624112882	+0.0742568382	8.870736454
38	118.5961766309	2.0740706879	−0.0732667027	8.864906647
39	121.7377420880	2.0854252422	+0.0723151467	8.859229272
40	124.8793089132	2.0964904866	−0.0713997282	8.853696559

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
41	128.0208770059	2.1072807979	+0.0705182163	8.848301319
42	131.1624462752	2.1178095078	−0.0696685682	8.843036885
43	134.3040166383	2.1280890011	+0.0688489095	8.837897066
44	137.4455880203	2.1381308034	−0.0680575164	8.832876097
45	140.5871603528	2.1479456587	+0.0672928009	8.827968605
46	143.7287335737	2.1575435988	−0.0665532972	8.823169576
47	146.8703076258	2.1669340045	+0.0658376495	8.818474318
48	150.0118824570	2.1761256608	−0.0651446023	8.813878437
49	153.1534580192	2.1851268069	+0.0644729905	8.809377815
50	156.2950342685	2.1939451798	−0.0638217315	8.804968583
51	159.4366111643	2.2025880546	+0.0631898176	8.800647102
52	162.5781886689	2.2110622805	−0.0625763097	8.796409948
53	165.7197667480	2.2193743133	+0.0619803313	8.792253893
54	168.8613453692	2.2275302450	−0.0614010632	8.788175891
55	172.0029245031	2.2355358310	+0.0608377387	8.784173063
56	175.1445041219	2.2433965140	−0.0602896398	8.780242689
57	178.2860842001	2.2511174463	+0.0597560927	8.776382192
58	181.4276647137	2.2587035104	−0.0592364646	8.772589131
59	184.5692456406	2.2661593371	+0.0587301608	8.768861190
60	187.7108269600	2.2734893229	−0.0582366213	8.765196170
61	190.8524086526	2.2806976452	+0.0577553186	8.761591984
62	193.9939907001	2.2877882769	−0.0572857554	8.758046644
63	197.1355730857	2.2947649996	+0.0568274620	8.754558260
64	200.2771557933	2.3016314151	−0.0563799947	8.751125031
65	203.4187388082	2.3083909573	+0.0559429339	8.747745239
66	206.5603221162	2.3150469021	−0.0555158823	8.744417247
67	209.7019057043	2.3216023771	+0.0550984638	8.741139490
68	212.8434895599	2.3280603705	−0.0546903214	8.737910475
69	215.9850736715	2.3344237388	+0.0542911166	8.734728774
70	219.1266580280	2.3406952151	−0.0539005280	8.731593019
71	222.2682426191	2.3468774157	+0.0535182499	8.728501903
72	225.4098274349	2.3529728465	−0.0531439918	8.725454173
73	228.5514124661	2.3589839095	+0.0527774772	8.722448626
74	231.6929977040	2.3649129085	−0.0524184425	8.719484113
75	234.8345831404	2.3707620540	+0.0520666369	8.716559527
76	237.9761687673	2.3765334684	−0.0517218210	8.713673807
77	241.1177545773	2.3822291905	+0.0513837662	8.710825933
78	244.2593405633	2.3878511803	−0.0510522546	8.708014926
79	247.4009267187	2.3934013221	+0.0507270777	8.705239844
80	250.5425130370	2.3988814292	−0.0504080363	8.702499779

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
81	253.6840995122	2.4042932472	+0.0500949399	8.699793860
82	256.8256861386	2.4096384570	−0.0497876061	8.697121244
83	259.9672729106	2.4149186784	+0.0494858602	8.694481124
84	263.1088598231	2.4201354726	−0.0491895350	8.691872717
85	266.2504468710	2.4252903453	+0.0488984701	8.689295272
86	269.3920340498	2.4303847494	−0.0486125117	8.686748061
87	272.5336213547	2.4354200870	+0.0483315122	8.684230384
88	275.6752087815	2.4403977121	−0.0480553299	8.681741563
89	278.8167963262	2.4453189327	+0.0477838286	8.679280944
90	281.9583839846	2.4501850127	−0.0475168778	8.676847897
91	285.0999717532	2.4549971743	+0.0472543516	8.674441808
92	288.2415596282	2.4597565990	−0.0469961292	8.672062089
93	291.3831476063	2.4644644304	+0.0467420942	8.669708166
94	294.5247356841	2.4691217748	−0.0464921346	8.667379487
95	297.6663238584	2.4737297040	+0.0462461428	8.665075516
96	300.8079121264	2.4782892553	−0.0460040147	8.662795734
97	303.9495004850	2.4828014340	+0.0457656504	8.660539638
98	307.0910889315	2.4872672144	−0.0455309532	8.658306742
99	310.2326774632	2.4916875409	+0.0452998301	8.656096573
100	313.3742660776	2.4960633298	−0.0450721913	8.653908673
101	316.5158547720	2.5003954693	+0.0448479502	8.651742598
102	319.6574435444	2.5046848218	−0.0446270230	8.649597916
103	322.7990323923	2.5089322243	+0.0444093289	8.647474210
104	325.9406213135	2.5131384890	−0.0441947898	8.645371073
105	329.0822103059	2.5173044056	+0.0439833303	8.643288109
106	332.2237993677	2.5214307406	−0.0437748773	8.641224937
107	335.3653884968	2.5255182391	+0.0435693603	8.639181183
108	338.5069776912	2.5295676252	−0.0433667110	8.637156486
109	341.6485669493	2.5335796034	+0.0431668633	8.635150492
110	344.7901562692	2.5375548582	−0.0429697534	8.633162861
111	347.9317456494	2.5414940560	+0.0427753191	8.631193258
112	351.0733350881	2.5453978449	−0.0425835005	8.629241359
113	354.2149245839	2.5492668559	+0.0423942396	8.627306850
114	357.3565141352	2.5531017031	−0.0422074799	8.625389422
115	360.4981037404	2.5569029846	+0.0420231669	8.623488778
116	363.6396933985	2.5606712829	−0.0418412476	8.621604625
117	366.7812831078	2.5644071655	+0.0416616706	8.619736681
118	369.9228728672	2.5681111852	−0.0414843861	8.617884667
119	373.0644626753	2.5717838810	+0.0413093457	8.616048316
120	376.2060525309	2.5754257783	−0.0411365025	8.614227364

TABLE I. (Continued)

s	$x_0^{(s)}$	$\log x_0^{(s)}$	$J_1[x_0^{(s)}]$	$\log \pm J_1[x_0^{(s)}]$
121	379.3476424328	2.5790373893	+0.0409658109	8.612421556
122	382.4892323800	2.5826192136	−0.0407972266	8.610630640
123	385.6308223712	2.5861717385	+0.0406307066	8.608854375
124	388.7724124055	2.5896954395	−0.0404662091	8.607092521
125	391.9140024818	2.5931907804	+0.0403036936	8.605344848
126	395.0555925990	2.5966582142	−0.0401431204	8.603611128
127	398.1971827563	2.6000981830	+0.0399844514	8.601891141
128	401.3387729527	2.6035111185	−0.0398276490	8.600184671
129	404.4803631872	2.6068974423	+0.0396726770	8.598491506
130	407.6219534590	2.6102575659	−0.0395195001	8.596811442
131	410.7635437673	2.6135918920	+0.0393680838	8.595144276
132	413.9051341111	2.6169008134	−0.0392183947	8.593489813
133	417.0467244898	2.6201847145	+0.0390704003	8.591847860
134	420.1883149024	2.6234439708	−0.0389240687	8.590218230
135	423.3299053483	2.6266789495	+0.0387793690	8.588600738
136	426.4714958268	2.6298900094	−0.0386362712	8.586995206
137	429.6130863371	2.6330775018	+0.0384947459	8.585401458
138	432.7546768784	2.6362417700	−0.0383547646	8.583819321
139	435.8962674503	2.6393831501	+0.0382162993	8.582248629
140	439.0378580519	2.6425019709	−0.0380793229	8.580689217
141	442.1794486827	2.6455985539	+0.0379438088	8.579140923
142	445.3210393420	2.6486732140	−0.0378097313	8.577603591
143	448.4626300292	2.6517262595	+0.0376770652	8.576077067
144	451.6042207439	2.6547579922	−0.0375457858	8.574561198
145	454.7458114852	2.6577687076	+0.0374158692	8.573055839
146	457.8874022529	2.6607586950	−0.0372872920	8.571560843
147	461.0289930463	2.6637282380	+0.0371600312	8.570076070
148	464.1705838648	2.6666776142	−0.0370340647	8.568601380
149	467.3121747080	2.6696070956	+0.0369093705	8.567136638
150	470.4537655754	2.6725169490	−0.0367859274	8.565681710

TABLE II.

THE FIRST 150 ROOTS OF $J_1(x)$ WITH CORRESPONDING VALUES OF $J_0(x)$

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
1	3.8317059702	0.5833921757	−0.4027593957	9.605045681
2	7.0155866698	0.8460639942	+0.3001157525	9.477288791
3	10.1734681351	1.0074690286	−0.2497048771	9.397427025
4	13.3236919363	1.1246245823	+0.2183594072	9.339171907
5	16.4706300509	1.2167102124	−0.1964653715	9.293286014
6	19.6158585105	1.2926073202	+0.1800633753	9.255425387
7	22.7600843806	1.3571738678	−0.1671846005	9.223196271
8	25.9036720876	1.4133613337	+0.1567249863	9.195138240
9	29.0468285340	1.4630987210	−0.1480111100	9.170294315
10	32.1896799110	1.5077166581	+0.1406057982	9.148003230
11	35.3323075501	1.5481720021	−0.1342112403	9.127788890
12	38.4747662348	1.5851759898	+0.1286166221	9.109297099
13	41.6170942128	1.6192717536	−0.1236679608	9.092257199
14	44.7593189977	1.6508834702	+0.1192498120	9.076457703
15	47.9014608872	1.6803487586	−0.1152736941	9.061730211
16	51.0435351836	1.7079407452	+0.1116704969	9.047938448
17	54.1855536411	1.7338835151	−0.1083853489	9.034970580
18	57.3275254379	1.7583631957	+0.1053740554	9.022733694
19	60.4694578453	1.7815360748	−0.1026005671	9.011149761
20	63.6113566985	1.8035346583	+0.1000351468	9.000152613
21	66.7532267341	1.8244722636	−0.0976530158	8.989685660
22	69.8950718375	1.8444465556	+0.0954333390	8.979700119
23	73.0368952256	1.8635423032	−0.0933584533	8.970153648
24	76.1786995846	1.8818335547	+0.0914132722	8.961009255
25	79.3204871755	1.8993853729	−0.0895848220	8.952234435
26	82.4622599144	1.9162552327	+0.0878618760	8.943800472
27	85.6040194364	1.9324941569	−0.0862346634	8.935681873
28	88.7457671449	1.9481476480	+0.0846946348	8.927855900
29	91.8875042517	1.9632564558	−0.0832342730	8.920302190
30	95.0292318080	1.9778572186	+0.0818469379	8.913002436
31	98.1709507308	1.9919829969	−0.0805267394	8.905940115
32	101.3126618230	2.0056637255	+0.0792684317	8.899100266
33	104.4543657913	2.0189265960	−0.0780673254	8.892469301
34	107.5960632595	2.0317963811	+0.0769192140	8.886034837
35	110.7377547809	2.0442957136	−0.0758203116	8.879785565
36	113.8794408476	2.0564453260	+0.0747672005	8.873711120
37	117.0211218989	2.0682642573	−0.0737567865	8.867801987
38	120.1627983281	2.0797700333	+0.0727862602	8.862049406
39	123.3044704886	2.0909788222	−0.0718530644	8.856445295
40	126.4461386985	2.1019055715	+0.0709548658	8.850982183

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
41	129.5878032451	2.1125641276	−0.0700895302	8.845653149
42	132.7294643885	2.1229673418	+0.0692551013	8.840451769
43	135.8711223648	2.1331271628	−0.0684497820	8.835372069
44	139.0127773887	2.1430547201	+0.0676719183	8.830408488
45	142.1544296559	2.1527603970	−0.0669199848	8.825555833
46	145.2960793452	2.1622538953	+0.0661925720	8.820809257
47	148.4377266203	2.1715442942	−0.0654883757	8.816164219
48	151.5793716314	2.1806401023	+0.0648061865	8.811616466
49	154.7210145163	2.1895493043	−0.0641448816	8.807162008
50	157.8626554019	2.1982794036	+0.0635034167	8.802797092
51	161.0042944054	2.2068374597	−0.0628808191	8.798518190
52	164.1459316346	2.2152301230	+0.0622761818	8.794321978
53	167.2875671898	2.2234636653	−0.0616886575	8.790205319
54	170.4292011631	2.2315440083	+0.0611174540	8.786165254
55	173.5708336411	2.2394767493	−0.0605618292	8.782198984
56	176.7124647027	2.2472671842	+0.0600210877	8.778303862
57	179.8540944227	2.2549203290	−0.0594945768	8.774477380
58	182.9957228701	2.2624409390	+0.0589816830	8.770717160
59	186.1373501093	2.2698335268	−0.0584818292	8.767020948
60	189.2789762004	2.2771023782	+0.0579944721	8.763386600
61	192.4206011997	2.2842515672	−0.0575190996	8.759812079
62	195.5622251597	2.2912849701	+0.0570552283	8.756295447
63	198.7038481298	2.2982062777	−0.0566024018	8.752834860
64	201.8454701561	2.3050190072	+0.0561601888	8.749428559
65	204.9870912823	2.3117265128	−0.0557281809	8.746074867
66	208.1287115488	2.3183319956	+0.0553059917	8.742772184
67	211.2703309942	2.3248385127	−0.0548932546	8.739518981
68	214.4119496545	2.3312489858	+0.0544896223	8.736313798
69	217.5535675637	2.3375662093	−0.0540947647	8.733155236
70	220.6951847537	2.3437928575	+0.0537083686	8.730041961
71	223.8368012552	2.3499314908	−0.0533301360	8.726972691
72	226.9784170965	2.3559845630	+0.0529597833	8.723946199
73	230.1200323046	2.3619544264	−0.0525970408	8.720961311
74	233.2616469051	2.3678433375	+0.0522416513	8.718016896
75	236.4032609225	2.3736534628	−0.0518933698	8.715111873
76	239.5448743794	2.3793868825	+0.0515519623	8.712245201
77	242.6864872978	2.3850455956	−0.0512172058	8.709415881
78	245.8280996983	2.3906315240	+0.0508888870	8.706622952
79	248.9697116003	2.3961465162	−0.0505668022	8.703865490
80	252.1113230227	2.4015923515	+0.0502507566	8.701142605

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
81	255.2529339830	2.4069707427	−0.0499405637	8.698453440
82	258.3945444983	2.4122833400	+0.0496360453	8.695797172
83	261.5361545844	2.4175317339	−0.0493370302	8.693173004
84	264.6777642566	2.4227174575	+0.0490433548	8.690580170
85	267.8193735296	2.4278419898	−0.0487548620	8.688017931
86	270.9609824173	2.4329067582	+0.0484714011	8.685485573
87	274.1025909327	2.4379131406	−0.0481928275	8.682982407
88	277.2441990888	2.4428624679	+0.0479190024	8.680507768
89	280.3858068975	2.4477560258	−0.0476497924	8.678061013
90	283.5274143702	2.4525950573	+0.0473850692	8.675641520
91	286.6690215183	2.4573807640	−0.0471247098	8.673248689
92	289.8106283521	2.4621143085	+0.0468685953	8.670881938
93	292.9522348815	2.4667968154	−0.0466166118	8.668540705
94	296.0938411167	2.4714293739	+0.0463686494	8.666224446
95	299.2354470668	2.4760130382	−0.0461246021	8.663932633
96	302.3770527404	2.4805488296	+0.0458843682	8.661664756
97	305.5186581464	2.4850377379	−0.0456478493	8.659420320
98	308.6602632927	2.4894807224	+0.0454149506	8.657198846
99	311.8018681874	2.4938787130	−0.0451855807	8.654999867
100	314.9434728378	2.4982326121	+0.0449596513	8.652822935
101	318.0850772513	2.5025432949	−0.0447370774	8.650667609
102	321.2266814346	2.5068116110	+0.0445177767	8.648533467
103	324.3682853946	2.5110383851	−0.0443016697	8.646420095
104	327.5098891378	2.5152244180	+0.0440886797	8.644327093
105	330.6514926702	2.5193704875	−0.0438787324	8.642254073
106	333.7930959978	2.5234773496	+0.0436717561	8.640200656
107	336.9346991265	2.5275457389	−0.0434676814	8.638166475
108	340.0763020616	2.5315763695	+0.0432664410	8.636151173
109	343.2179048084	2.5355699358	−0.0430679701	8.634154402
110	346.3595073722	2.5395271133	+0.0428722055	8.632175826
111	349.5011097577	2.5434485591	−0.0426790865	8.630215115
112	352.6427119699	2.5473349127	+0.0424885539	8.628271950
113	355.7843140132	2.5511867967	−0.0423005506	8.626346020
114	358.9259158923	2.5550048172	+0.0421150209	8.624437021
115	362.0675176112	2.5587895644	−0.0419319113	8.622544658
116	365.2091191742	2.5625416132	+0.0417511694	8.620668644
117	368.3507205852	2.5662615238	−0.0415727448	8.618808700
118	371.4923218481	2.5699498420	+0.0413965883	8.616964550
119	374.6339229664	2.5736071000	−0.0412226523	8.615135931
120	377.7755239442	2.5772338166	+0.0410508905	8.613322583

TABLE II. (Continued)

s	$x_1^{(s)}$	$\log x_1^{(s)}$	$J_0[x_1^{(s)}]$	$\log \pm J_0[x_1^{(s)}]$
121	380.9171247846	2.5808304976	−0.0408812580	8.611524252
122	384.0587254911	2.5843976364	+0.0407137112	8.609740691
123	387.2003260668	2.5879357143	−0.0405482076	8.607971661
124	390.3419265151	2.5914452012	+0.0403847061	8.606216926
125	393.4835268389	2.5949265553	−0.0402231666	8.604476258
126	396.6251270412	2.5983802242	+0.0400635503	8.602749432
127	399.7667271248	2.6018066446	−0.0399058191	8.601036230
128	402.9083270927	2.6052062433	+0.0397499364	8.599336438
129	406.0499269472	2.6085794367	−0.0395958663	8.597649849
130	409.1915266912	2.6119266320	+0.0394435740	8.595976259
131	412.3331263272	2.6152482268	−0.0392930254	8.594315469
132	415.4747258576	2.6185446099	+0.0391441877	8.592667285
133	418.6163252847	2.6218161610	−0.0389970285	8.591031516
134	421.7579246109	2.6250632515	+0.0388515167	8.589407978
135	424.8995238385	2.6282862444	−0.0387076217	8.587796488
136	428.0411229696	2.6314854948	+0.0385653138	8.586196870
137	431.1827220064	2.6346613498	−0.0384245640	8.584608949
138	434.3243209507	2.6378141492	+0.0382853441	8.583032555
139	437.4659198047	2.6409442255	−0.0381476266	8.581467523
140	440.6075185704	2.6440519037	+0.0380113847	8.579913690
141	443.7491172494	2.6471375021	−0.0378765921	8.578370897
142	446.8907158436	2.6502013323	+0.0377432234	8.576838988
143	450.0323143550	2.6532436992	−0.0376112537	8.575317810
144	453.1739127852	2.6562649015	+0.0374806587	8.573807214
145	456.3155111358	2.6592652316	−0.0373514146	8.572307055
146	459.4571094085	2.6622449759	+0.0372234984	8.570817188
147	462.5987076049	2.6652044151	−0.0370968875	8.569337473
148	465.7403057266	2.6681438239	+0.0369715599	8.567867774
149	468.8819037749	2.6710634717	−0.0368474939	8.566407955
150	472.0235017515	2.6739636224	+0.0367246686	8.564957885

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FRANK NELSON COLE

BY T. S. FISKE

Frank Nelson Cole, son of Otis Cole and Frances Maria Pond Cole, was born at Ashland, Massachusetts, September 20, 1861. Prepared for college at the Marlborough (Massachusetts) High School, he was admitted to Harvard in September 1878 and at graduation in 1882 stood second in his class. After studying mathematics for a year in the Graduate School of Harvard University, he was appointed to a traveling fellowship and spent the following two years (1883–1885) at the University of Leipsic, where he was a member of the mathematical seminar of Professor Felix Klein.

Returning to Harvard in 1885, he was for two years lecturer in the Graduate School. In 1886 he received from Harvard the degrees of A.M. and Ph.D. In the summer of 1888 he revisited Germany and married Miss Martha Marie Streiff of Hamburg.

In the autumn of 1888 he became instructor in mathematics at the University of Michigan, and a year later was promoted to an assistant professorship. In 1895 he was called to Columbia University as professor of mathematics with assignments to the University Faculty of Pure Science and to the Faculty of Barnard College. For fourteen years (1901–1914) he served on the Committee on Admissions of Barnard College. During three years (1904–1906) he was an examiner in mathematics for the College Entrance Examination Board.

He was elected Secretary of the American Mathematical Society in 1895, became a member of the editorial staff of the Bulletin in 1898, and succeeded the author of this sketch as editor-in-chief of the Bulletin in 1899. In 1920 he relinquished his administrative and editorial labors for the American Mathematical Society. However, he continued his work in the Columbia University Faculty of Pure Science and at Barnard College.

In the spring of 1926 he applied to Columbia University to be released from active service on September 20, 1926, the sixty-fifth anniversary of his birth. Suddenly taken seriously ill he died of heart failure May 26, 1926. He was survived by his wife, two sons, and a daughter.

A picture of Frank Nelson Cole in 1885 after his return from Leipsic is sketched in the following excerpt from an address "Professor Bôcher's Scientific Start in Life" delivered by Professor William Fogg Osgood at a meeting of the Harvard Mathematical Club held February 12, 1925:

> "The other course, Mathematics 13, was given by Mr. (at that time not yet Dr.) Frank Nelson Cole, who had just returned from Germany and was aglow with the enthusiasm which Felix Klein inspired in his students. Cole was not the first to give a formal course of lectures at Harvard on the theory of functions of a complex variable, Professor James Mills Peirce having lectured on this subject in the seventies. That presentation was, however, solely from the Cauchy

standpoint, being founded on the treatise of Briot and Bouquet, *Fonctions Elliptiques*. Cole brought home with him the geometric treatment which Klein had given in his noted Leipsic lectures of the winter of 1881–1882. Cole also gave a course in Modern Higher Algebra, with its applications to geometry. The enthusiasm which he felt for his subject was contagious. Interesting as were the other courses I have mentioned, they stood as the Old over against the New and of the latter Cole was the apostle. The students felt that he had seen a great light. Nearly all the members of the Department,—Professor J. M. Peirce, Dr. B. O. Peirce, and I think, Professor Byerly,—attended his lectures. It was the beginning of a new era in in graduate instruction at Harvard, and mathematics has been taught here in that spirit ever since."

In the publication of scientific papers the most productive period of Professor Cole's life was the seven years (1888–1895) during which he was connected with the University of Michigan. He had previously published three papers, one of them a long critique of Klein's *Ikosaeder*, in the American Journal of Mathematics. While at Michigan he published eight papers and a translation of the work of Eugen Netto on *The Theory of Substitutions*. In the period of thirty-one years during which he was connected with Columbia University he published nine scientific papers, over one hundred reports of meetings of the American Mathematical Society, and innumerable unsigned notes containing items of interest to the mathematical community.

From the time of his arrival in New York in 1895, Cole devoted the larger part of his time, thought, and efforts to the administrative and editorial activities of the American Mathematical Society. His greatest and most absorbing interest in life seemed to be that the affairs of the American Mathematical Society should be conducted wisely, efficiently, and with good taste. He was tireless in his opposition to anything that seemed to him slovenly, unscholarly, or born of an excess of zeal.

In 1920, upon the conclusion of a quarter of a century of service to the Society, some hundreds of his fellow members joined in presenting him with a purse as a token of their admiration and gratitude. This he turned back to the Society, and upon the recommendation of the Council it was used for the purpose of founding a prize to be known as the Frank Nelson Cole prize in Algebra. At the same time in recognition of his distinguished services volume 27 of the Bulletin (the thirtieth volume counting from the establishment of the Bulletin in 1891) was dedicated to him. His portrait appears as the frontispiece of that volume.

At the meeting of the Mathematical Society held September 9, 1926, the following minute was adopted:

"On the recommendation of its Council, the American Mathematical Society, meeting in Columbus, Ohio, on September 8, 1926, expresses its deep sense of loss in the death of Frank Nelson Cole on May 26, 1926. He was for many years the Society's most active executive officer. From an early date in its history until 1920 when he passed his duties on to others, he ably guided the development of the

Society. As Secretary from 1895, and as Editor-in-Chief of the Bulletin from 1899, he led the Society from its modest beginnings to a state of solid accomplishment. He exercised his functions with a skill which excited admiration and which gave the American Mathematical Society an established place in the scientific world. When he retired he turned over to his successors a healthy structure that was able to withstand the stresses of the very difficult post-war period.

The Society wishes now to place on record its grateful recognition of his devoted service to the ideals of American Mathematical Science. His memory will remain an inspiration to all who may in the future serve the interests of the Society and the cause of mathematics in America."

In order to realize how valuable and timely was the service Cole rendered to mathematicians by publishing his translation of Netto's *Theory of Substitution Groups*, one must notice that it was practically the first textbook on the subject in English. European students had easy access to Groups in Serret's *Algèbre Superieure* from the 3rd edition (1866), and in the German edition of Netto's book (1882). Yet no English author had at that date taken up the task of presenting the subject for beginners; if we except seven pages in Chrystal's *Algebra*, and journal papers by Cayley and Bolza. An additional value attaches to Cole's book, in that it gave Netto opportunity to revise, omit, and recast considerable portions of the German text.

A very definite contribution to group theory is his work on simple groups, in two papers, one with Glover's cooperation. These showed only three simple groups of orders between 200 and 661. Burnside and Dickson later carried the upper boundary to 1000.

The papers cited below on triad systems and their groups mark an interesting outgrowth of the Netto translation. Debate arose over a matter left indefinite in that book, whether every triad system must have a transitive group. A survey of the literature showed examples to the contrary, one of the two possible triad systems on thirteen elements having a group of order six. The possibility of enumerating all the triad systems on fifteen elements was then discussed, whereupon Miss Cummings developed the earliest examples of incongruent systems exhibiting congruent groups, and extended the number of known systems to twenty-four. Mr. White followed this with a complete synthesis of all having actual groups bringing the number up to forty-four. Cole meantime had begun, but deferred the completion of, an exhaustive discussion of the structure of such systems, based on "interlacings." When he was shown some novel systems, which admit no group of substitutions save the identity, a race began for results. The two others found a way of deriving new systems from old, and had produced thirty-three of this new sort, before Cole finished his investigation showing about fifty such, in addition to those admitting groups. Miss Cummings, applying to all these her speedy method of comparison by "indices," discovered many duplicates. Cole had not overlooked any that were already known, but had found thirty-six "groupless" systems, the last three being entirely new. This brought the whole list up to eighty; and

Cole's deductive method furnished the climax, the proof that now the list is complete. This was probably the only investigation in which he collaborated with others; and even in this his methods were entirely independent, only the results permitting comparison and control.

The short paper on Kirkman Parades was essentially an extract from this larger work, and served to settle conclusively the "fifteen schoolgirl problem."

List of Professor Cole's Publications

Reports of Meetings of the American Mathematical Society: this Bulletin, vol. 3, p. 1; vol. 4, pp .1, 87, 175, 291, 415; vol. 5, pp. 1, 113, 217, 325, 423; vol. 6, pp. 95, 117, 267, 365; vol. 7, pp. 1, 113, 199, 289; vol. 8, pp. 1, 183, 367; vol. 9, pp. 183, 281, 393, 525; vol. 10, pp. 53, 119, 171, 221, 373, 485; vol. 11, pp. 111, 231, 347, 461; vol. 12, pp. 53, 107, 223, 325, 423; vol. 13, pp. 55, 153, 261, 315, 423; vol. 14, pp. 53, 157, 247, 351, 407; vol. 15, pp. 157, 275, 323, 419; vol. 16, pp. 53, 169, 281, 395, 451; vol. 17, pp. 61, 167, 277, 389, 505; vol. 18, pp. 51, 159, 213, 327, 485; vol. 19, pp. 51, 163, 275, 387, 497; vol. 20, pp. 169, 283, 395, 507; vol. 21, pp. 57, 161, 269, 373, 481; vol. 22, pp. 161, 263, 373, 481; vol. 23, pp. 57, 157, 257, 299, 435; vol. 24, pp. 57, 169, 265, 369, 465; vol. 25, pp. 49, 145, 433; vol. 26, pp. 145, 241, 337, 433; vol. 27, p. 97.

The potential of a shell bounded by confocal ellipsoidal surfaces, Proceedings of the American Academy of Arts and Sciences, vol. 9 (1883), pp. 226–231.

A contribution to the theory of the general equation of the sixth degree, American Journal of Mathematics, vol. 8 (1885–86), pp. 265–286.

Klein's Ikosaeder, American Journal of Mathematics, vol. 9 (1886–87), pp. 45–61.

On rotations in space of four dimensions, American Journal of Mathematics, vol. 12 (1889–1890), pp. 181–210.

The linear functions of a complex variable, Annals of Mathematics, vol. 5, (1889–90), pp. 121–176.

Simple groups from order 201 *to order* 500, American Journal of Mathematics, vol. 14 (1891), pp. 378–388.

Review of Felix Klein, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, Bulletin of the New York Mathematical Society, vol. 1 (1891–92), p. 105.

Translation of Eugen Netto, *The Theory of Substitutions and its Applications to Algebra*. Ann Arbor, 1892.

Note on the substitution groups of six, seven, and eight letters, Bulletin of the New York Mathematical Society, vol. 2 (1892–93), pp. 184–190.

The transitive substitution groups of nine letters, Bulletin of the New York Mathematical Society, vol. 2 (1892–93), pp. 250–258.

On groups whose orders are products of three prime factors (with J. W. Glover), American Journal of Mathematics, vol. 15 (1893), pp. 191–220.

Simple groups as far as order 600, American Journal of Mathematics, vol. 15 (1893), pp. 303–315.

Review of Charlotte Angas Scott, *An Introductory Account of Certain Modern Ideas in Plane Analytical Geometry*, Bulletin of this Society, vol. 2 (1895–96), pp. 265–271.

On the factoring of large numbers, Bulletin of this Society, vol. 10 (1903–04), pp. 134–137.

The groups of order $p^\alpha q^\beta$, Transactions of this Society, vol. 5 (1904), pp. 214–219.

The triad system of thirteen letters, Transactions of this Society, vol. 14 (1913), pp. 1–5.

Note on solvable quintics, This Bulletin, vol. 21 (1914–15), pp. 462–464.

The complete enumeration of triad systems in fifteen elements (with L. D. Cummings and H. S. White), Proceedings of the National Academy, vol. 3 (1917), pp. 197–199.

Complete classification of the triad systems on fifteen elements (with L. D. Cummings and H. S. White), Memoirs of the National Academy, vol. 14, No. 2 (1919); 89 pp.

Kirkman Parades, Bulletin of this Society, vol. 28 (1922), pp. 435–437.

On simple groups of low order. Bulletin of this Society, vol. 30 (1924), 489–492.

HAUSDORFF'S REVISED MENGENLEHRE

Mengenlehre. By Felix Hausdorff. Second edition. Berlin and Leipzig, Walter de Gruyter, 1927. 285 pp.

There are second editions and second editions. Some are merely second printings with misprints corrected and a few pages added to bring the subject matter up to date; others are complete revisions. Hausdorff's second edition is an extreme example of the latter type; here even the title has been revised the first edition having appeared in 1914 under the title *Grundzüge der Mengenlehre.**

Not only has the title of the book been abbreviated in this new edition, but the book itself has been reduced in size so that the number of pages is considerably less than two-thirds that of the first edition. This is due principally to a difference in the literary style of the two editions, the *Grundzüge* seeming extremely verbose when compared with the conciseness of the second edition. It is questionable whether the new edition will be as "teachable" as the old. Certainly it cannot be read with as much ease by a student approaching the subject for the first time, as so much more is left for him to supply for himself. Probably the best approach to the subject will be found to lie in a judicious selection of topics from both editions. We regret that we find in the second edition such a conscious effort on the part of the author to save space.

As in the first edition, there are two main topics considered, the first five chapters being concerned with the general theory of aggregates and the remaining chapters with the more special theory of point sets. In discussing the various chapters we shall give in general only the omissions from and the additions to the material contained in the first edition.

The *Vorbemerkungen* and Chapter 1 (pp. 9–24) contain the essential parts of the first two chapters of the *Grundzüge* in one-third the space required there. Among the topics omitted are symmetric sets and the principle of duality, and much less space is devoted to the "algebra" of sets. Among the topics retained the order is often changed; for example, a section on functions defined over a set is placed before the section on the fundamental ideas of the sum and common part† of two sets. Here the author has replaced the cumbersome $\mathfrak{S}$, $\mathfrak{D}$ notation of the *Grundzüge* by a more common notation: the sum S of two sets A, B is indicated by $S=A+B$, the common part D by $D=AB$. As in the *Grundzüge*, the notation $S=A+B$ is used if and only if $D=0$, thus differing from the usual notation which indicates the sum by $S=A+B$ in all cases.

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* Reviewed by Professor Henry Blumberg in this BULLETIN, vol. 27 (1920–21), pp. 116–129.

† *Durchschnitt*, translated by Blumberg as *section*. The most common expression is *product*, but Hausdorff uses this in a different sense.

Chapter 2 (pp. 25–41) on cardinal numbers and Chapter 3 (pp. 41–55) on order types correspond almost exactly to Chapters 3 and 4 respectively of the *Grundzüge*, although the number of pages has been reduced by half. This is partly accounted for by the omission of certain sections on ordered sets, the topic of ordered sets being given very brief consideration in this edition.

The fourth chapter (pp. 55–77) contains all the material on well-ordered sets and ordinal numbers that has been retained from Chapters 5 and 6 of the *Grundzüge*, where these topics covered five times the space. Practically all of Chapter 6 (partially-ordered sets, etc.) has been omitted, the only section that remains essentially as a unit being the section on general products and powers of ordered sets. The other topics considered are: the well-ordering theorem, the comparability of ordinal numbers, combination of ordinal numbers, and the Alephs.

In the fifth chapter (pp. 77–93) the author begins a comprehensive treatment of Borel and Suslin (or Souslin) sets, to which portions of Chapters 7 and 8 are also devoted. Suslin sets were first studied in 1917 by Michael Suslin (1894–1919), a Russian. These sets are a generalization of the well known Borel sets.

The remainder of the book (see however §40) is devoted to a discussion of sets of points considered as subsets of a metric space, that is, a space in which there exists a definition of the distance between each pair of points, subject to certain distance axioms. This is a departure from the viewpoint of the *Grundzüge*, where Hausdorff begins with a topological space, i.e., a space satisfying certain neighborhood axioms (*Umgebungsaxiome*). Having proved a number of theorems on the basis of this set of axioms, he then specializes the space step by step by adding additional restrictions, obtaining in turn a metric space, a euclidean space of n dimensions, and finally a euclidean plane, with appropriate groups of theorems in each case.

We do not quarrel with the author's decision to omit the discussion of euclidean spaces from the second edition, although we feel that the student loses something if he does not have brought to his attention the special methods applicable in euclidean spaces.

We do regret exceedingly, however, that he has treated only metric spaces when he might so easily have followed the outline of the *Grundzüge*. Instead of doing so, he proceeds as follows: having assumed that the space is metric, he defines (§22) a *neighborhood* of radius ρ of a point x, as the set of points whose distance from x is less than ρ. A system of neighborhoods defined thus will satisfy the *Umgebungsaxiome*. In proving many of the theorems that follow, the author makes use of this special system of neighborhoods, and in his proofs does not use all the properties of a metric space, but only those properties which are necessary to insure that the space contain a system of neighborhoods satisfying the *Umgebungsaxiome*. In other words, a number of theorems which are true in a topological space are proved by Hausdorff only for a metric space, when with a slight rearrangement of material he could have proved these theorems in all their generality. Since the author does not hesitate to assume whenever

necessary that his given metric space is complete, compact, or separable, if one of these hypotheses is needed to prove a particular theorem, it seems strange that he does not take the most general viewpoint throughout this part of the book, and assume merely that the point sets considered are subsets of a *topological* space, assuming it to be also metrical when and only when it is necessary. In this connection, compare §22 and §23 of the second edition with §2 and §3 of Chapter 7 of the first edition.

Chapter 6 (pp. 94–164) on point sets and Chapter 7 (pp. 164–193) on point sets and ordinal numbers correspond to Chapters 7 and 8 of the *Grundzüge*. Dyadic sets and sets of the first and second categories of Baire are given more prominence than in the first edition. A few pages are devoted to the idea of a locally connected (=connected *im kleinen*) set, an idea which Hahn and Mazurkiewicz were developing at the time of the appearance of the first edition. The results of the sixth chapter are obtained without any reference to ordinal numbers, while Chapter 7 is devoted to theorems in which this concept occurs or which are more easily proved by making use of this idea. Here the author shows that the totality of Suslin sets is not identical with the totality of Borel sets, but that there exist Suslin sets which are not Borel sets. In the concluding section, necessary and sufficient conditions are given in order that a Suslin set be a Borel set. In an appendix are given alternative proofs by Lusin of the theorems of this section, which proofs were communicated to the author by letter too late to be put in the proper place in the text. This is a striking proof of the current interest in this topic.

Chapter 8 (pp. 193–232) on correspondences between two spaces and Chapter 9 (pp. 232–275) on real functions comprise some material from Chapter 9 of the *Grundzüge*, but much of the material is new. The author first considers the subject of continuous curves: it is shown that the class of dyadic continua is identical with that of continuous curves; the Sierpinski and the Hahn-Mazurkiewicz characterizations of continuous curves are given. Next the subject of correspondences is taken up, under which are Lavrientieff's theorem on the extension of a continuous (1–1) correspondence, and a convenient table (p. 219) giving the image of various classes of point sets under correspondences of various types. Another section is devoted to the work of Hahn and R. L. Moore on the prime parts of a continuum. In the final section of Chapter 8 (§40), Hausdorff discusses briefly topological spaces, giving various sets of axioms used to distinguish the class of metric spaces from the more general class of topological spaces.

The ninth and concluding chapter gives a much more complete treatment of Baire's functions than was the case in the *Grundzüge*, and concludes with the theorems of Hahn and Sierpinski on the nature of the convergence set of a sequence of real continuous functions.

In addition to the omissions which we have mentioned above, the second edition omits the entire tenth chapter of the *Grundzüge* which treated content and measure of point sets and the applications of this theory to Lebesgue integrals.

The book closes with a list of the literature of the subject, a list of sources, and an index. Unlike the first edition, the list of sources gives

merely the original place of publication of the theorems cited; there are no supplementary remarks. The enlarged list of literature over that given in the first edition testifies to the growing interest in the subject of aggregate theory—an interest which the appearance of the *Grundzüge* did much to arouse and foster. The index is remarkably complete, and is an improvement over that of the first edition in that it contains also references to authors. Another improvement is the consecutive numbering of sections. The work of the publishers is excellent. We have discovered only a few unimportant misprints.

It is not within our province to criticize the author's choice of material. The book aims to be a textbook and not a report on the subject of aggregate theory, and therefore many topics have had to be omitted altogether, and in the case of those topics which have been included a choice of theorems has had to be made. Under such circumstances, the only criterion for the goodness of the author's choice is the taste of the individual reader.

The *Grundzüge* has been out of print for the past four years and we therefore heartily welcome the appearance of this new edition. We wish to state without qualification that this is an indispensable book for all those interested in the theory of aggregates and the allied branches of real variable theory and analysis situs. If we have seemed critical of some phases of the book, it is only because this excellent book is not better.

H. M. GEHMAN

GENERAL MATHEMATICS IN EUROPE

Leçons de Mathématiques générales. By L. Zoretti. With preface by P. Appell. Second edition. Paris, Gauthier-Villars, 1925. 16+788 pp.

Nouveau Traité de Mathématiques générales. By C. Fabry. Fourth edition. Vol. 2. Paris, Hermann, 1925. 276 pp.

Einführung in die Elemente der höheren Mathematik. By H. Hahn and H. Tietze. Leipzig, Hirzel, 1925. 12+331 pp.

Cours Complet de Mathématiques Spéciales. By J. Haag. Vol. 1, *Algèbre et Analyse.* 1914. 6+402 pp. Vol. 2, *Géométrie.* 1921. 7+662 pp. Vol. 3, *Mécanique.* 1922. 8+192 pp. Paris, Gauthier-Villars.

In the United States the term "general mathematics" has been used recently to denote a course for high school students. Such a course may have many advantages but it presents a troublesome problem to educational authorities when a pupil seeks entrance to college. It is difficult to determine whether certain definite entrance requirements have been met and it is sometimes equally puzzling to know whether the student is qualified to take certain courses in the college curriculum. General mathematics in a high school may consist of a mixture of arithmetic, algebra, geometry, and trigonometry in all sorts of proportions. The purpose usually is to prepare the pupils for practical living and college preparation is not the chief aim of the course.

On the continent of Europe the term "general mathematics" has been used to denote a course corresponding more nearly to those given in

American colleges. The course consists of a mixture of analysis, geometry, and mechanics, the emphasis on different topics varying considerably with the authors. The purpose is to furnish the mathematical equipment required by physicists, chemists, engineers, and other scientists and the training of mathematicians is not the chief aim of the course.

The first edition of the book by Zoretti was well reviewed by Professor J. B. Shaw in this Bulletin (April, 1918, p. 355). The second edition differs from the first only in a few details, notably in an increase of the use of graphical methods and in the omission of the chapter on elliptic functions.

The book by Fabry, of which only the second volume is at hand, is now in its fourth edition, from which we may infer that the book has been successful in meeting the need for which it was designed. This volume includes analysis, mechanics, and theory of errors.

The book by Hahn and Tietze covers much less ground than the others and the point of view is that of the pure mathematician. The four parts into which the book is divided are: 1. Numbers; 2. Functions; 3. Differential Calculus; 4. Integral Calculus. The first two parts give an exposition of the elements of the theory of functions of real variables, furnishing a firm foundation for the introduction to the calculus which follows.

The work by Haag is more extensive than any of the others. It is based on a course given by the author preparing students for entrance to the École Polytechnique and the École Normale Supérieure, but has been extended considerably beyond this course with the idea of giving adequate preparation for advanced work in mathematics. Apparently this accounts for the title "special mathematics." A student of the so-called general mathematics would be expected to omit certain parts.

The complete work consists of four volumes. In addition to the three listed above, volume 4 has been announced with the title "Géométrie descriptive. Trigonométrie." Each volume of text is accompanied by a separate volume of exercises, some of which are worked out while others are left to the student. The author lays great stress upon the exercises as necessary for a complete understanding of the subject matter and has provided a very generous collection. Any student who would follow the author's advice and work through the volumes of exercises after studying the corresponding text, could not fail to be well grounded in the mathematics usually covered in America through the first year of the graduate school.

W. R. LONGLEY

REYMOND ON SCIENCE IN ANTIQUITY

History of the Sciences in Greco-Roman Antiquity. By Arnold Reymond. Translated from the French by Ruth Gheury de Bray, with preface by Léon Brunschvicg. New York, E. P. Dutton. x+246 pp. $2.50.

In his preface to this work M. Brunschvicg calls attention to the fact that Professor Reymond has for many years given lectures on the history of science in the University of Neuchatel. These lectures have been attended both by the students of the Faculté des Lettres and by those of the Faculté des Sciences, and similar provision has of late been made in the University of Lausanne. These facts, while having little to do with the merits of the book under review, are significant as regards the interest being shown in European universities in the history of science as a culture subject.

The work treats of the mathematical, astronomical, physical, and natural sciences, and what will here be said concerning the treatment of the first of these will probably allow for a fair estimate of the treatment of the others.

It is apparent from the first chapter that M. Reymond writes as a philosopher rather than as a scientist or a historian. He has gone to few original sources but has depended chiefly upon Zeuthen, Loria, Tannery, and Ball, with four references to Cantor and two to Heath and a like number to Montucla's work of more than a century and a quarter ago. Ball, who is hidden in the index under "Rouse Ball," is referred to more often than Zeuthen or Loria, while Tropfke is unknown. Even the frequent references to such men as Euclid and Archimedes show that the author's knowledge comes rather from writers like Boyer and Ball than from any study of the classics which these men wrote.

As a result of this dependence upon secondary sources, often of a rather inferior type, the work has many errors that will at once occur to anyone who is at all familiar with the history of mathematics. For example, it is not likely that any historian of the subject would care to subscribe to statements such as these:

"As to the information furnished by hieroglyphics and cuneiforms, it amounts to little." To be sure M. Reymond could assert that the Rhind Papyrus is in hieratic and not hieroglyphic, but the impression given by the statement is unfortunate, particularly in view of what the cuneiform tablets are revealing as to the mathematics of Babylon.

"We are reduced for the most part to conjectures concerning the scientific knowledge of the Egyptians and Chaldeans." On the contrary, we have very positive acquaintance with a considerable range of such knowledge as witness, in the case of Egypt, Professor Archibald's extensive bibliography in Dr. Chace's edition of the Rhind Papyrus, now in press.

"In practice and for reckoning they (the Egyptians and Chaldeans) made use of abacuses the arrangement of which calls to mind the ball-

frame formerly used in infants' schools." Aside from a single reference in Herodotus we have no direct reference to the Egyptian use of the abacus, and that goes back only to the fifth century B.C., more than a thousand years after Ahmes wrote—and as to the Chaldeans we have no direct evidence whatever.

"It would appear also that the Egyptians, as well as the Hindoos, had discovered, before Pythagoras, the relation between the surfaces of squares constructed on the sides of a right-angled triangle." It would be most interesting to have some proof for such a sweeping assertion as this.

These statements are merely typical. In the space allowed to this review it is not feasible to mention others.

As to dates, if the reader places any dependence upon those given, he will be led into difficulties. When M. Reymond states that Thales lived "about 624–528 B.C.," he is on safe ground; that is, he asserts that the dates are merely approximate. On the other hand there is no satisfactory evidence for saying that Pythagoras "died in the year 500 B.C.," that Zeno was "twenty-five years younger than his master" (Parmenides), the dates for neither being known with any degree of precision; that Hippocrates was born in 470 B.C., or Eudoxus in 408 B.C., or Euclid in 330 B.C., and so for various other such positive assertions. One of the most certain dates of this kind in Greek history is that of the birth of Archimedes, but this is here given as 257 B.C., when it should be 287.

There are various other types of error. For example, the assertion that Bryson contributed notably to the method of exhaustion (p. 56) is now considered very uncertain; that "the quadrature of the circle is, as we know, an insoluble geometrical problem" (p. 58) is a misleading assertion unless "geometrical" is defined before this statement is made; that the *Method* of Archimedes was discovered on a palimpsest "of Jerusalem" (pp. 75, 76) instead of "in Constantinople"; that the "writings" of Diophantus "became known" to the scientific world through Regiomontanus (p. 98), especially as M. Reymond himself says later (pp. 126, 127) that "it was only in the year 1575 that he (Diophantus) became known to the western world."

The bibliography will be helpful to students of the history of science in spite of such misprints as *uber* for *über*, "Kultur *de* Gegenwart," and J. H. Heiberg for J. L. Heiberg.

The index contains merely names. It is of no value to anyone wishing to look up topics. Even under names the reader will look in vain for Peet, Rhind, Pistelli, Festa, Friedlein, Hoche, Ilberg, and various others mentioned in the text or bibliography. He will, however, find both men bearing the name of Diocles listed under the same item and, as elsewhere stated, he will look unsuccessfully for Ball unless he happens to think of W. W. Rouse Ball.

On the whole the work has merits as a philosophical dissertation but none as a source of accurate historical information in the field of mathematics.

DAVID EUGENE SMITH

REVISION OF PASCH'S GEOMETRY

Vorlesungen über neuere Geometrie. By Moritz Pasch. Second edition. With an appendix: *Die Grundlegung der Geometrie in historischer Entwicklung* by Max Dehn. (Die Grundlehren der mathematischen Wissenschaften, Band 23.) Berlin, Springer, 1926. x+275 pp.

In 1882, Pasch's *Vorlesungen über neuere Geometrie* inaugurated a new era in the development of projective geometry. For the first time, projective geometry was made a purely deductive science founded upon and logically developed from a complete system of postulates. At the same time, since the postulates completely ignored the concept of parallelism, it was definitely made independent of euclidean or any metric geometry. This was in accord with a previous discovery by Klein that the theorems of projective geometry are independent of the parallel postulate.

The book was reprinted in 1912 with a short appendix and in 1913 it was translated into Spanish and the appendix of the 1912 edition was distributed over the book as smaller appendices at the ends of different sections.

The second edition just published follows the plan of the Spanish edition. The portion revised is not extensive. The rewritten material totals about two pages in the second edition and takes the place of about five pages of the first edition. It consists almost entirely of simplified proofs. In one place (page 47, second paragraph) a proof that had covered two pages is now given in one short paragraph.

The second edition contains a total of eight pages of added material. This is inserted piecemeal as supplements to the various sections or as isolated paragraphs between two formerly adjacent paragraphs. Some new theorems are added, for example: p. 7, Kernsatz IX; p. 25, Lehrsätze 13, 14, 15; p. 39, theorem by Ventura Reyes y Prosper; p. 76, Sätze J and P. Aside from these, the supplementary material consists mainly of comments and references.

The typographical errors of the first edition are corrected. This was not done in the reprint of 1912.

The evident desire to supplement rather than revise this important pioneer work is best explained by the author himself in his preface to the second edition. After discussing some of the more recent methods and results, he continues, "I do not desire to incorporate these new geometric concepts in the new edition, but prefer to keep the point of view of the first edition. If the reader will allow himself to be carried back in imagination for half a century, he may wish at least to acquaint himself with the geometry of that period and with the position this book had in its time in the foundations of geometry."

As the preferable alternative to enlarging and revising his own book so as to include the new concepts, Pasch requested Dehn to write an appendix covering this field. The last eighty-seven pages of the book

comprise this appendix which, as stated in the preface, corresponds to a two-hour semester course of lectures given by the author.

In the introduction, the development of the investigations in the foundations of geometry is divided into four periods:

1. The parallel postulate of Euclid.
2. The continuity postulate of Archimedes.
3. Projective geometry.
4. The modern period in which the foundations of geometry have been studied as such. This period began with the work of Lobatchewsky in 1826.

The first chapter gives a rapid survey of the attempts to prove the euclidean parallel postulate, the new geometrical concepts that were initiated by these attempts, and especially the development of these concepts from the time of Lobatchewsky to that of Pasch.

The four remaining chapters follow closely in content and method the work of Hilbert contained in his *Grundlagen der Geometrie* (1899) and in a later memoir in the Mathematische Annalen (1903). Many references to Hilbert assign to him the authorship of the major portion of these chapters. More recent authors are also frequently cited, but most of these citations refer to new derivations or extensions of Hilbert's results. Some of these extensions were made by Dehn himself, notably his work in pseudo-geometry and non-Archimedean geometry.

The contributions of certain other geometers, particularly E. H. Moore and Veblen, should have had a place in this supplement. The findings of E. H. Moore that some of Hilbert's axioms are redundant and that Schur's statement concerning the redundance of certain others is incorrect are especially pertinent.

Some interesting new features are: The serviceable table on page 229 shows distinctly the interrelations of various basic theorems of projective geometry. The extended principle of duality is proved (pp. 237–239) to be a direct consequence of the commutative law for the multiplication of line segments. The euclidean parallel postulate and the Archimedean postulate of continuity are shown to have a projective property in common (pp. 247–248).

The appendix is ably written and well worth reading. Only the work of Hilbert is given in detail, but most of the principal European geometers are cited. The author has succeeded remarkably well in presenting such an extensive subject so briefly.

It is peculiarly fitting that these two contributions to the foundations of geometry should be bound in one volume—the classical work of Pasch, reëdited after a half century by himself and the great work of Hilbert, rewritten, extended at times and revised somewhat in methods of presentation and proof by one of his former students who was with him when these concepts were taking form.

T. R. HOLLCROFT

FOUR BOOKS ON VECTOR ANALYSIS

1. *Vectorrechnung.* By Jean Spielrein. 2d edition. Stuttgart, Konrad Wittwer, 1926. xvi+434 pp.
2. *Die Vektoranalysis und ihre Anwendung in der theoretischen Physik,* Teil 1: *Die Vektoranalysis.* Teil 2: *Anwendung der Vektoranalysis in der theoretischen Physik.* By W. v. Ignatowsky. 3d edition. Leipzig and Berlin, Teubner, 1926. x+110 pp.+iv+123 pp.
3. *Die ebene Vectorrechnung und ihre Anwendungen in der Wechselstromtechnik*; Teil 1, *Grundlagen.* By Heinrich Kafka. Leipzig and Berlin, Teubner, 1926. viii+132 pp.
4. *Initiation aux Méthodes Vectorielles et aux Applications Géométriques de l'Analyse.* By G. Bouligand and G. Rabaté. Paris, Librairie Vuibert, 1926. viii+215 pp.

A course in vector analysis is now regarded as a necessary part of the training of any student of physics or of applied mathematics. Indeed no one can intelligently study any branch of mathematics without soon meeting the concept which is at the foundation of the whole subject of vector analysis, namely the concept of invariance under certain specified types of transformations. It is the great merit of the modern treatment of tensor analysis that the allowable transformations are kept as general as possible; for most purposes differentiability and reversibility being sufficient. Looked at from this point of view tensor analysis is a *general* vector analysis in the sense of E. H. Moore. It seems, then, to the present reviewer a little unfortunate that the writers of texts on vector analysis still treat in such detail the special vector analysis where the allowable transformations are those from one set of rectangular Cartesian coordinates to another. No doubt the reason given would be a pedagogic one but very often a view from a higher standpoint gives a real understanding that is almost impossible to secure from a lower level.

The first of the books under review is encyclopaedic in character, and can be highly recommended as a reference work rather than as a text. Some idea of its completeness may be gathered from the fact that appended to the book is a "Formelsammlung" in which are listed no less than 881 of the principal results. A rather severe attack of mental indigestion would threaten an immature student who tried to learn his subject from this book. A very valuable feature of the book, which is carefully printed and bound, is a collection of 190 problems with detailed solutions. The following chapter headings will give an idea of the contents.

Elementare Vektoroperationen. Besondere Vektoren. Bezeichnungen. Funktionen skalarer Veränderlichen. Ortsfunktionen. Geometrie der Vektorfelder. Lineare Vektorfunktionen. Affinoralgebra. Affinoranalysis.

The books 2 and 3 are, respectively, nos. 6 and 22 of the valuable series of monographs, known as the "Sammlung Mathematisch-Physikali-

scher Lehrbücher," which is being published under the general editorship of E. Trefftz. Ignatowsky's work is now in its third edition and is well and favorably known. It is not intended to be nearly as complete as Spielrein's work to which it refers for more advanced parts of the subject. The first edition (1909–1910) was reviewed in this Bulletin (vol. 17, pp. 102–104). The changes in the third edition are of a minor character. For a student interested mainly in physical applications this book is one of the best on the subject. Kafka's book is intended principally for students of electrical engineering. Here consideration is restricted to plane vectors and the subject is merely a geometrical consideration of complex variable theory. The electrical engineer has by this time a pretty definitely established mode of treatment; this is very different, at least in matters of detail, from the established method of the mathematician. The present work should be very useful and easy reading for a student beginning the study of functions of a complex variable.

The book by Bouligand and Rabaté is very different from the three already mentioned, reflecting a national difference of emphasis. Here the interest is almost entirely geometrical and the treatment is clear and elementary. We would recommend the work to a student who contemplates a study of Darboux's monumental work on the theory of surfaces, or of Appell's treatise on mechanics. The following chapter headings give a good idea of the contents.

Vecteurs. Produit scalaire et applications. Produit vectoriel. Théorie des moments; systèmes de vecteurs glissants. La dérivation géométrique et la théorie des courbes. Étude intrinsèque des fonctions de point. Problèmes d'intégration aux point de vue géométrique. Théorie des surfaces.

F. D. MURNAGHAN

SHORTER NOTICES

Die Welt der Atome. By Arthur Haas. Berlin, Walter de Gruyter & Co., 1926. xii+130 pp.

The evolution of our ideas with respect to physical phenomena has been so rapid during the last few years that one who does not devote a fair amount of time to following the development of the subject is likely to feel confused and fail to realize the intimate connection between the new results of experiment. Dr. Haas gave, during the summer of 1926, a series of ten lectures at the University of Vienna, which were intended to give the general public a comprehensive view of the modern physics of the atom, and these lectures have now been published. The author has succeeded admirably in giving a very clear presentation of the modern conception of the atom and of atomic phenomena; and even many who have followed more or less closely the recent developments will find a reading of this book to be both interesting and profitable. In view of the purpose of the author, it is natural that he should have described a very definite model of an atom, and perhaps not laid sufficient emphasis upon the provisional character of the model. But most of the discoveries that have recently been made have resulted from a very definite picture of the atom and so this can hardly be considered a serious fault.

E. P. ADAMS

Mathematical Investigations in the Theory of Value and Prices. By Irving Fisher. New Haven, Yale University Press, 1926. xii+126 pages.

This interesting little book is a photo-engraved reprint of the memoir by Irving Fisher on the subject published in 1892 and long since out of print. An eight page review of the book was published by Professor Fiske in this Bulletin (vol. 2 (1893), pp. 204–211). The impressions of the present reviewer in relation to the book were well expressed by Professor Fiske. In spite of the fact that certain theorists are at present inclined to the view that the students of economics have very commonly been on the wrong track in their efforts to adopt the methods of the physical sciences, it seems fairly obvious that Professor Fisher's book clarifies certain economic theories by the use of mathematics and physical analogies. Thus, clearness is added to the idea of marginal utility by introducing a unit of measurement for utility and by establishing relations among marginal utility, total utility, utility-value, and gain at a given time.

That the book has been recognized by mathematical economists as an enduring contribution to the theory of value and prices is fairly obvious in the light of the facts that it was translated into French by Jacques Moret twenty-five years after the first publication, and that there existed a demand that it be reprinted a third of a century after its first publication.

H. L. RIETZ

The Byzantine Astrolabe at Brescia. By O. M. Dalton. New York, Oxford University Press (American Branch). 14 pp.+3 plates. Price 70 cents.

This monograph was communicated to the British Academy on July 7, 1926, and is now reprinted from the Proceedings. The astrolabe described in the paper has an unusual interest because it is of Byzantine workmanship and dates from a period of which, as regards the science of astronomy on the banks of the Bosphorus, we know but little. The instrument is larger than most of those which have come down to us, being nearly fifteen inches in diameter, but the workmanship is crude in comparison with the neat work of European craftsmen of the Middle Ages or with the Arabic pieces. A special interest, however, lies in the fact that the inscriptions are all in Greek, the instrument being, as the author states, "perhaps the sole survival of an astrolabe representing a Greek type apparently unmodified by foreign ideas." On the arachne are some iambic verses which state that the piece,

> "being an intricate work, with ardent mind fashioned Sergius the Persian, holding a consul's rank";

and another inscription reads

> "Decree and command of Sergius, protospatharios, consul, and man of science [?], in the month of July, fifteenth indiction, year 6570,"

being the year 1062 of our era.

The author prefaces his monograph with a brief statement as to the general nature of an astrolabe and then gives description of the one which is the subject of his researches. The instrument is now in the Museo dell'Età Cristiana at Brescia, having been presented in 1844. The essay is a genuine contribution to our knowledge not only of the history of the astrolabe but of the state of science in the eleventh century.

DAVID EUGENE SMITH

Les Groupes Abéliens Finis et les Modules de Points Entiers. By Albert Chatelet. Paris and Lille, 1925. 221 pp.

As the author states in the introductory chapter, this book was the outgrowth of a much more limited treatment of abelian groups that he had originally planned as the first part of a treatise on abelian algebraic fields. While much more narrow in scope than the existing texts on groups, which treat those of the commutative sort more or less incidentally, it neverthe less embraces considerable material that is not found in such texts and, while following in large part the developments of the subject as given in papers by various authors, appears nevertheless to have considerable novelty in its treatment.

When the elements of an abelian group are represented in all possible ways as products of powers of a suitably chosen set, the set of exponents of such powers may be regarded as the coordinates of points in a certain number of dimensions. The set of points that thus correspond to the identical element of the abelian group form a "module" of a particular type in the same space, i.e., the "sum" or "difference" of any two points

in the set is itself in the set. More generally, any element of the abelian group corresponds to a set of points with integral coordinates that are "congruent" with respect to this module. Following the method employed by Frobenius and Stickelberger in their well known paper on abelian groups, the author therefore develops their properties very largely from the corresponding properties of modules.

The problem of making a change in the "basis" of a module, i.e., a finite number of points in terms of which all the points of the module may be represented by means of integral multipliers, involves naturally the question of the equivalence of matrices of a somewhat more restricted sort than that employed in the usual algebraic theory. The proof of the existence of a "reduced" basis is omitted (rather unfortunately, it seems to the reviewer), reference being made to another text by the same author. The questions of "divisibility," "greatest common divisor," and "least common multiple" of modules and matrices are treated quite fully. The same is true of the subject of the "invariants" of matrices with integral elements, which bears rather a close analogy to the more familiar theory of elementary divisors.

Following the rather extended treatment of modules and matrices in the second chapter, the remainder of the book is devoted to abelian groups. The subsequent chapters vary considerably in interest and importance, the subject of the automorphisms of an abelian group being treated quite comprehensively, while the theory of group characters, being included in the corresponding theory for finite groups in general, seems too special to attract the reader particularly. This would have been unavoidable, however, unless the scope of the book had been considerably widened.

The book should prove valuable as a reference text on the subject of which it treats, although too specialized probably to have a very wide use. It has evidently been very carefully prepared, including as it does a complete index of definitions and a fairly comprehensive bibliography. In the text definitions are emphasized by bold-faced type and theorems are in italics. Some use of numbered formulas might have been an improvement.

H. H. MITCHELL

Höhere Mathematik. Teil I. *Differentialrechnung und Grundformeln der Integralrechnung, nebst Anwendungen.* By Rudolf Rothe. 2d edition. Leipzig, B. G. Teubner, 1927. vii+186 pp.

The first edition of this volume was reviewed in this Bulletin in vol. 31 (1925), pp. 566–67. The second edition differs from the first only in unessential details, mainly matters of rewording in a number of places. One error which crept into the first edition might well have been corrected in the second; viz., on p. 116 the author states that the equations $x-e^z\cos y=0$, $y-e^z\sin y=0$ have but one common solution. By sketching the curves it is easily evident that they have an infinity of solutions, which fact one might easily guess by considering the equivalent equation $e^z=z$ in the complex variable.

T. H. HILDEBRANDT

Cambridge Readings in the Literature of Science. Being Extracts from the Writings of Men of Science to Illustrate the Development of Scientific Thought. Arranged by W. C. D. and M. D. Whetham. Cambridge, The University Press, 1924. x+275 pp.

One of the most difficult things for a student of history to do is to reconstruct the "mental furniture" of other times and civilizations—to comprehend feelings and mental attitudes which were natural products of an environment containing elements and circumstances so foreign to the present-day student as to be entirely unsuspected by him.

It is in this respect that the reading of source material is most helpful. By a study of the exact language of original documents one gains, not only more precise and accurate information, but also something of the spirit and intellectual perspective of the time, which summaries and secondary accounts can not supply.

In the last twenty-five years, especially, these facts have become well recognized and many collections of "Select Documents," "Select Readings from the Original Sources," "Reprints and Translations," etc., have appeared. Most of these collections have been in the field of social sciences. The book under review is a welcome addition to the relatively few which have been published in the field of natural sciences.

Mr. Whetham and his daughter have arranged a selection of extracts from the writings of men of science to illustrate the development of scientific thought along three lines:

(1) the structure of the universe—cosmogony;
(2) the nature of matter—atomic theories;
(3) the development of life—evolution.

Along these lines the authors "try to trace the thoughts from the inspired poetry of the Book of Genesis to the latest revelations of the telescope and the laboratory."

There is a satisfaction in reading Newton's ideas about the *System of the World*, and John Dalton's *On Chemical Synthesis* exactly as they formulated them. Translations of extracts from the writings of Aristotle, Archimedes, Copernicus, Laplace, Lamarck, Bergson, and more than a dozen others, are a bit less satisfying than if given in the original language, but are, of course, much more widely available. The insertion of explanatory and historical material between extracts aids greatly in carrying the thread of the story and these insertions seem to the reviewer to be very well written.

In looking over any such collection one is likely to feel that there are omissions more notable than some of the selections given and it is quite probable that many readers of the book under consideration will have that impression. For example, how many will feel that a worthy attempt at tracing the development of human thought concerning the structure of the universe can afford to give no quotation from the great *Almagest* of Claudius Ptolemaeus, the greatest existing authority on ancient astronomy, or can afford, in tracing the story of the theory of evolution, to omit some quotation from Alfred Russell Wallace's famous essay *On the Tendency*

of Varieties to Depart Indefinitely from the Original Type, which led to the publication of Darwin's preliminary essay in 1858? However, the matter of selection is always a difficult one and largely dependent upon individual judgments.

The value of the book, in the opinion of the reviewer, would have been greatly enhanced by incorporating with each extract a paragraph* indicating various other publications, if such exist, where the writing from which the extract was taken (or the original text in case of a translation) may be found. To quote an extract as "From *The Works of Archimedes*, edited by Sir Thomas Heath," with no mention of publishers or date or place of publication or reference as to where the original Greek text may be found, and no listing of cross-references to other publications containing the same writing, offers little help and encouragement to one who may be interested in investigating the subject more fully.

U. G. MITCHELL

Les Nouveaux Axiomes de l'Electronique. By R. Ferrier. Paris, A. Blanchard, 1925. 61 pp.

This is an attempt to modify the laws of electrodynamics to fit the quantum theory of Bohr. The author considers two point charges and puts the energy of the system equal to the quotient of a function of the relative velocity by the distance between the charges. This leads to a fundamental dynamical frequency proportional to the energy instead of proportional to the three-halves power of the energy as in the classical dynamics. While the potential energy on the classical theory is inversely proportional to the distance between charges, it is difficult to see how any theory which hoped to obtain macroscopic results at all comparable with experiment can proceed from an expression for the total energy of the form used in this monograph.

LEIGH PAGE

The Quantum Theory. By N. M. Bligh. New York, Longmans, Green & Co., 1926. 112 pp. $3.00.

Beginning with the general problem of radiation, the author has given a very brief sketch of the development of the quantum theory leading to Planck's formula. He next introduces the light-quantum hypothesis and its application to the photo-electric effect. The subject of atomic heats is briefly considered, and finally Bohr's theory of spectra and atomic constitution is outlined. The treatment throughout is too little critical and too superficial to be of much value, but it may serve a useful purpose as an outline, with references to the original sources, of the older form of the quantum theory.

E. P. ADAMS

* Such, for example, as the excellent bibliographical paragraphs in William MacDonald's *Select Documents Illustrative of American History*, New York, The Macmillan Company, 1924.

Traité de Physique. Tome supplémentaire: *La Physique de* 1914 *à* 1926, première partie. By O. D. Chwolson. Translated by A. Corvisy. Paris, Hermann, 1927. 339 pp.

Investigations on the Theory of the Brownian Movement. By Albert Einstein. Translated by A. D. Cowper. New York, Dutton. viii+124 pp.

Three Lectures on Atomic Physics. By A. Sommerfeld. Translated by H. L. Brose. New York, Dutton. 70 pp.

The first of these three volumes (translated from the Russian) contains a rather elementary account of the progress in physics from 1914 to 1926. It is divided into nine chapters as follows: the charge and the mass of the electron (chapter I), theory of quanta (II), the structure of the atom (III and IV), line spectra (V), X-rays (VI), band spectra (VII), ultra-violet and infra-red rays (VIII), excitation and ionization of gases by electrons (IX). The treatment is mainly physical in character, and the mathematical analysis is reduced to a minimum. Each chapter is followed by a bibliography indicating the sources of the material and giving references to further related matters.

During the period from 1905 to 1911 Einstein set forth the results of his investigations on the theory of the Brownian movement. Concerning this work of Einstein, F. A. Lindemann (in his article on Einstein, Enc. Brit., 13th edition) says: "One of his [Einstein's] earliest publications gave the complete theory and formulae of the phenomenon known as Brownian motion, which had puzzled physicists for nearly 80 years. He showed that the heat motion of particles, which is too small to be perceptible when these particles are large, and which can not be observed in molecules since these themselves are too small, must be perceptible when the particles are just large enough to be visible, and gave complete equations which enable the masses themselves to be deduced from the motions of these particles." The second volume in the foregoing list contains five of Einstein's most important papers on the Brownian movement together with a substantial appendix of notes by R. Fürth, containing corrections, later references, and some expansions of the original text. The book will serve as an introduction to the study of the Brownian movement.

In his London University lectures on atomic physics (third volume in the foregoing list), Sommerfeld undertakes to set forth his "present opinions" on current problems of atomic physics. He deals mainly with recent developments of the quantum theory. In the first lecture he presents among other things a new theory of the hydrogen atom according to which it has a structure analogous to that of the alkali metals. The second lecture is entitled "The general system of the complex terms." In the third lecture the author deals with the periodic system of the elements and the general problem of chemical linkage. The lectures are of interest to students of physics rather than of mathematics.

R. D. CARMICHAEL

NOTES

The April, 1927, number of the American Journal of Mathematics (volume 49, No. 2) contains the following papers: *A class of functions harmonic within the sphere*, by H. E. Bray and G. C. Evans; *Notes on formal modular protomorphs*, by O. C. Hazlett; *Irreducible continuous curves*, by H. M. Gehman; *The plane quintic with five cusps*, by M. Lehr; *The convergence of general means and the invariance of form of certain frequency functions*, by E. L. Dodd; *On the interpolatory properties of a linear combination of continuous functions*, by D. V. Widder; *Note on Tschebycheff approximation*, by D. V. Widder; *Generalization of Waring's theorem on fourth, sixth, and eighth powers*, by L. E. Dickson; *On complete systems of irrational invariants of associated point sets*, by C. M. Huber; *Regular maps and their groups*, by H. R. Brahana; *The groups belonging to a linear associative algebra*, by A. Ranum.

The July, 1927, number of the Annals of Mathematics (series 2, volume 28, No. 3) contains: *Sextic surfaces with a double septimic curve*, by B. C. Wong; *Determination of a basis for the integral elements of certain generalized quaternion algebras*, by M. D. Darkow; *Expansions in Bessel functions*, by M. H. Stone; *On the Green's function for systems of differential equations*, by W. M. Whyburn; *On Lamé families of surfaces*, by C. E. Weatherburn; *Derivation of the Fredholm theory from a differential equation of infinite order*, by H. T. Davis; *Asymptotic expression for the probability of trials connected in a chain*, by E. D. Pepper; *On certain indefinite quaternary forms representing all integers*, by C. G. Latimer; *Application of algebraic number theory to congruences involving binomial coefficients*, by H. S. Vandiver; *Ternary quadratic forms and congruences*, by L. E. Dickson; *Correspondences between algebraic curves*, by S. Lefschetz; *On the existence and properties of the solutions of a certain differential equation of the second order*, by T. H. Gronwall; *A new definition of almost periodic functions*, by N. Wiener; *A theory of integers, in relation to the iteration of algebraic functions*, by O. E. Glenn; *On systems of total differential equations*, by J. M. Thomas; *On positive solutions of a system of linear equations*, by L. L. Dines; *On completely signed sets of functions*, by L. L. Dines; *Concerning continuous curves and correspondences*, by W. L. Ayres.

The British Association for the Advancement of Science met at Leeds, August 31–September 7, 1927. Professor E. T. Whittaker presided over Section A (mathematical and physical sciences); his address was entitled *The outstanding problems of relativity.*

Cambridge University has awarded its Mayhew prize in applied mathematics to J. Hargreaves, of Clare College, and its Ricardo prize in thermodynamics to J. N. Goodier, of Downing College.

The James Scott prize of the Royal Society of Edinburgh, for the period 1922–1926, "for a lecture or essay on the fundamental concepts of natural philosophy," has been awarded to Sir Joseph Larmor.

Professors A. Einstein, of Berlin, O. Hölder, of Leipzig, F. Schur, of Breslau, and E. Study, of Bonn, have been elected corresponding members of the Bavarian Academy of Sciences.

Professor P. Koebe, of Leipzig, has been elected a member of the Saxon Academy of Sciences.

Dr. Fridjof Nansen, professor of oceanography at the University of Oslo, has been elected a corresponding member of the Prussian Academy of Sciences in the section of mathematical physics.

The Technical School at Darmstadt has conferred an honorary doctorate of engineering on Professor G. Scheffers, of the Berlin Technical School, on the occasion of his sixtieth birthday.

Professor Charles Fabry, director of the institute of optics of the University of Paris, has been elected a member of the Paris Academy of Sciences in the section of physics, as successor to the late Daniel Berthelot.

Professors N. Bohr and A. Einstein have been elected foreign honorary fellows of the Royal Society of Edinburgh.

On the occasion of the meeting of the British Association at Leeds, the University of Leeds conferred an honorary doctorate of science on Professor R. A. Millikan, of the California Institute of Technology.

Professor G. H. Hardy, of Oxford University, has been elected foreign associate of the National Academy of Sciences.

Professor J. S. Ames, of Johns Hopkins University, has been elected chairman of the National Advisory Committee on Aeronautics.

Professor F. R. Moulton has been elected a member of the executive board of the National Research Council.

Dr. Irwin Schrödinger, of the University of Zurich, has been called to the University of Berlin, as successor to Professor Max Planck.

The following have been admitted as private docents in German universities: F. Lettenmeyer, at Munich; B. L. van der Waerden, at Göttingen.

M. Bachelier, of the University of Rennes, has been appointed professor of the differential and integral calculus at the University of Besançon.

Dr. T. M. MacRobert, of the University of Glasgow, has been promoted to a professorship of mathematics.

Mr. L. A. Pars, of Jesus College, and Mr. H. A. Newman, of St. John's College, have been elected university lecturers in mathematics at Cambridge.

Mr. H. E. Arnold has been promoted to an assistant professorship at Wesleyan University.

Dr. Walter Bartke has been appointed assistant professor in mathematical astronomy at the University of Chicago.

Dr. R. W. Barnard has been appointed assistant professor of mathematics at the University of Chicago.

Dr. H. Y. Benedict, professor of applied mathematics and dean of the College of Arts and Sciences at the University of Texas, has been elected president of the University.

Mr. A. H. Blue, of the University of Iowa, has been elected professor of mathematics at West Union College.

Dr. C. C. Camp, of the University of Illinois, has been appointed associate professor of mathematics at the University of Nebraska.

Professor D. F. Campbell has resigned, and associate professor C. I. Palmer has been made professor of mathematics at the Armour Institute of Technology. Professor Palmer is also acting dean of students.

Mr. W. B. Campbell, of Cornell University, has been appointed assistant professor of mathematics at Colgate University.

Associate Professor Arnold Emch has been appointed professor of mathematics at the University of Illinois.

Assistant Professor W. L. Crum, of Harvard University, has been appointed professor of statistics in the Graduate School of Business, Stanford University.

Associate Professor H. J. Ettlinger has been appointed professor of mathematics at the University of Texas.

Professor H. S. Everett, of Bucknell University, has been appointed extension professor of mathematics at the University of Chicago.

Assistant Professor M. C. Foster, of Williams College, has been appointed associate professor of mathematics at Wesleyan University.

Assistant Professor W. V. N. Garretson, of Rutgers University, has been appointed head of the department of mathematics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor Jewell C. Hughes, of the University of Arkansas, has been appointed associate professor of mathematics there.

Dr. C. H. Langford, of Harvard University, has been appointed assistant professor of philosophy at the University of Washington.

Mr. W. T. MacCreadie, of Cornell University, has been appointed assistant professor of mathematics at Bucknell University.

Dr. L. H. MacFarlan has been appointed assistant professor of mathematics at the University of Washington.

Dr. Florence M. Mears, of Cornell University, has been appointed professor of mathematics at Alabama College.

Dr. E. B. Miller, of the University of Wisconsin, has been appointed professor of mathematics at Illinois College, Jacksonville.

Mr. H. C. Schaub, of Cornell University, has been appointed assistant professor of mathematics at Washington and Jefferson College.

Dr. Marion E. Stark, of Wellesley College, has been appointed assistant professor of mathematics there.

Dr. J. M. Thomas has been appointed assistant professor of mathematics at the University of Pennsylvania.

Dr. V. A. Tam has been appointed professor of mathematics at the University of the Phillipines.

Dr. W. J. Trjitzinsky has been appointed assistant professor of mathematics at the University of Valparaiso, Valparaiso, Indiana.

Assistant Professor L. A. H. Warren, of the University of Manitoba, has been appointed full professor of mathematics there.

Dr. Louis Weisner has been appointed assistant professor of mathematics at Hunter College.

Mr. P. D. Wilkins of Case School of Applied Science has been appointed assistant professor of mathematics at Bates College.

Dr. W. H. Wilson, of the University of Iowa, has been appointed associate professor of mathematics at the University of Florida.

The following appointments to instructorships in mathematics are announced:

University of Akron, Ohio, Mr. Samuel Silberfarb;
University of Arkansas, Mr. C. R. Worth;
University of Chicago, Dr. F. R. Bamforth;
Cornell University, Messrs. E. H. Hadlock, R. L. Jeffery, P. W. Swingle, C. C. Torrance, and F. G. Williams;
University of Florida, Mr. C. A. Messick;
Hunter College, Miss Mary V. Kenny, Dr. Louis Weisner;
Illinois State Teachers College, Miss Gladys H. Freeman;
Johns Hopkins University, Mr. Howard Pixley;
Northwestern University, Dr. Lois W. Griffiths;
Oklahoma Agricultural and Mechanical College, Mr. H. C. Billings;
Pennsylvaina State College, Dr. Marguerite Darkow, and Miss Marie Johnson;
University of Rochester, Dr. Raymond Garver;
Rutgers College, Messrs. M. G. Boyce and C. R. Wilson;
Stanford University, Miss Rosa L. Jackson;
Wells College, Miss Ethel I. Moody;
Junior College at Jefferson City, Miss Frances Baker.

Professor G. Mittag-Leffler, of the University of Stockholm, founder and editor of Acta Mathematica, died July 12, 1927, at the age of eighty-one.

Dr. Albert Gockel, professor of cosmic physics at the University of Freiburg, Switzerland, died March 4, 1927.

Dr. Anton Wassmuth, professor of mathematics at the University of Graz, has died, at the age of eighty-two.

Mr. A. C. Bose, controller of examinations at Calcutta University, died December 11, 1926. Mr. Bose had been a member of the American Mathematical Society since 1916.

Dr. William Burnside, formerly professor of mathematics at the Royal Naval College, Greenwich, died August 21, 1927, at the age of seventy-five.

Dr. I. B. Crandall, of the Bell Telephone Laboratories, died April 22, 1927, at the age of thirty-six.

Dr. G. A. Hill, senior astronomer of the Naval Observatory, died August 29, 1927, at the age of sixty-nine.

Professor H. D. Thompson, of Princeton University, has died, at the age of sixty-three. Professor Thompson had been a member of the American Mathematical Society since 1897.

NEW PUBLICATIONS

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THIRTY-SIXTH ANNUAL LIST OF PAPERS

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ST. LOUIS, MISSOURI, November 26, 1927. The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 5.

At the request of the Program Committee, Professor E. B. Stouffer will give an address entitled *Some canonical forms and associated expansions in projective differential geometry.*

NASHVILLE, TENNESSEE, ANNUAL MEETING, December 27-28, 1927.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than November 29. At the usual joint meeting with other organizations, Professor James Pierpont will represent this Society, and will give an address on *Mathematical rigor, past and present.* The fifth Josiah Willard Gibbs lecture, under the auspices of this Society and the American Association for the Advancement of Science, will be delivered by Professor E. W. Brown.

NEW YORK CITY, February 25, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than February 4.

R. G. D. RICHARDSON,
Secretary of the Society.

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